Elusive graph properties

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World Logic Day 2022

Joint work with with **Tamás Csernák**

Elusive properties of infinite graphs, arxiv note

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- finite combinatorics, theoretical computer science

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- 1. P is a non-trivial property
- 2. P is a graph property (i.e. preserved by isomorphism)

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- P is a non-trivial graph property. V is a (finite) vertex set.
- Test this property by asking questions of the form " is there an edge between vertices x and y?"
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A bold conjecture (Aanderaa-Rosenberg)

$\mu(P, n) = \binom{n}{2}$ *for any non-trivial graph property P and $n \in \omega$.*

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- Ask all the edges between A_0 and A_1. We know u.
- Ask all the pairs which contains u. We know the partition.
- Ask all the pairs $v_i v_j$ and $v_i x_k$. We know if G has property P_n.
- There is $k < 2$ and $i \neq j < 3$ such that $x_i, x_j \in A_k$. We did not asked $x_i x_j$.

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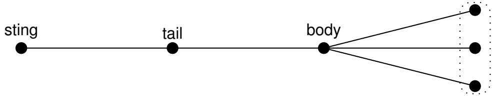

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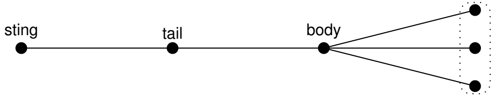

Theorem (Best,Boas, Lenstra)

There is an algorithm using only $O(n)$ questions to determine if G is a scorpion.

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A property P is ***elusive*** *if* $\mu(P, n) = \binom{n}{2}$*.*

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Theorem (Yao)

If P is a monotone, non-trivial graph property, then P is elusive on the bipartite graphs.

What about infinite graphs?

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- $\min G = \langle V, E \rangle$ and $\max G = \langle V, E \cup U \rangle$.

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(6) *the Hider **wins** iff the graph $\langle V, E_{G_\alpha} \rangle$ does not have property R, but graph $\langle V, E_{G_\alpha} \cup U_{G_\alpha} \rangle$ has property R for each $\alpha < \beta$.*

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A graph property R is ***elusive*** *on a set V iff the Hider has a winning strategy in the game $\mathbb{E}_{V,R}$.*

A positive theorem

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Theorem

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- Otherwise e_α will be a nonedge.

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The property " **every vertex has infinite degree**" is clearly monotone and has property $(*)$.

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Revised Naive conjecture: **Every "natural" monotone graph property is elusive on every infinite vertex set**.

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For each natural number n and for each infinite set V the monotone graph property R_n

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is not elusive on V.

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Actually, the Seeker has winning strategies in both cases.

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R is ***H*-elusive** *iff Hider has a winning strategy in $\mathbb{E}_{V,R}$.*

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Let $H = \langle V, E^* \rangle$ *be a graph. A vertex set* $L \subset V$ *is a* ***covering*** *set iff for each* $v \in V \setminus L$ *there is* $a \in L$ *with* $\{v, a\} \in E^*$*.*

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Proposition

Given an infinite cardinal κ, the infinite complete graph K_κ, the balanced bipartite graph $K_{\kappa,\kappa}$, the Turan graphs $T(\kappa, n)$ for $n \in \omega$, and the Cantor graph C are braided.

Theorem
For each natural number n and for each infinite braided graph $H = \langle V, E^* \rangle$ the monotone graph property R_n

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is not H-elusive.

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*A graph property R is **$\mathcal{I}$-elusive** on a set V iff Hider has a winning strategy in the game $\mathbb{E}_{V,\mathcal{I},R}$.*

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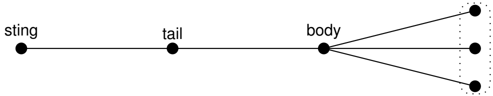

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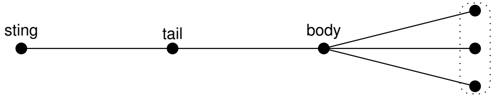

Theorem

(1) For each natural number n the monotone graph property R_n

$$\text{"}\deg_G(v) \geq n \textbf{ for each vertex } v\text{"}$$

is $\mathcal{I}_{SF}$-elusive on the vertex set ω.

The ideal $\mathcal{I}_{SF}$ of square-free sets.

$\mathcal{I}_{SF} = \{F \subset [\omega]^2 : \neg\exists W \in [\omega]^\omega \; [W]^2 \subset F\}$.

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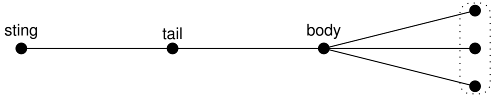

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(2) For each natural number n the monotone graph property C_n

$$\text{"}c_G(v) \geq n \textbf{ for each vertex } v\text{"}$$

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What about uncountable vertex sets?

G contains $K_{1,n}$

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Theorem

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*"G **contains** $K_{1,n}$"*

is $\mathcal{I}_n$-elusive on the vertex set ω, where

$$\mathcal{I}_n = \{E \subset [\omega]^2 : \neg\exists B \in [\omega]^n \; E \cup [B]^2 = [\omega]^2\}.$$

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  (Otherwise)$\{i, j\} \in U_\alpha \cap [A, \omega]$ with $i \in A$ implies $d_{E_\alpha \cup U_\alpha}(i) \geq n$.
- $P_\alpha \cup [B]^2 = [\omega]^2$ and so $P_\alpha \notin \mathcal{I}_n$.

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Decide if the the following property are elusive or not:

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- G ***contains*** K_ℓ

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- G ***contains*** P_3
- G ***contains*** K_ℓ
- G ***is connected***

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Decide if the the following property are elusive or not:

- *G **contains** P_3*
- *G **contains** K_ℓ*
- *G **is connected***

Find the right AKR-style statement/conjecture for infinite graphs