Strong downward Löwenheim-Skolem theorems, large cardinals and set-theoretic complexity

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Introduction

$$\frac{\textit{Vopěnka}}{\textit{supercompact}} = \frac{\textit{subtle}}{\textit{strongly unfoldable}}$$

The groundbreaking work of Cohen and Gödel on the *Continuum Hypothesis* revealed that many natural mathematical questions are not answered by the standard axiomatization of mathematics provided by the axioms of $\mathbf{ZFC}$.

This initiated the programme to search for intrinsically justified extensions of $\mathbf{ZFC}$ that settle these questions.

Among the axioms studied in this programme, large cardinal axioms play a special role.

These axioms typically postulate the existence of cardinals having certain properties that make them very large, and whose existence cannot be proved in $\mathbf{ZFC}$, because it implies the consistency of $\mathbf{ZFC}$ itself. But various other axioms are also considered to fall into this category.

Large cardinal axioms answer many important questions in a desirable way and this leads many set theorists to think that these axioms should be included in the correct axiomatization of mathematics.

In addition, the following two empirical facts explain the special role of large cardinals in the programme outlined above:

- First, there is strong evidence that for every extension of $\mathbf{ZFC}$, the consistency of the given theory is either equivalent to the consistency of $\mathbf{ZFC}$, or to the consistency of some extension of $\mathbf{ZFC}$ by large cardinal axioms.
- Second, all large cardinal notions studied so far are linearly ordered by their consistency strength.

In combination, these phenomena allow for an ordering of all mathematical theories in a linear hierarchy based on their consistency strength.

Despite their central role in modern set theory, large cardinals are still surrounded by many open conceptual questions:

- There is no widely accepted formal definition of the intuitive concept of large cardinals. Instead there are several common ways to formulate such principles (elementary embeddings, partition properties, etc.).
- Moreover, although the linearity of the ordering of mathematical theories by their consistency strength seems to be a fundamental fact of mathematics, it has not been possible to prove the general validity of this principle and, without a formal definition for the concept of large cardinals, it is not even clear how such a proof should look like.
- Finally, although large cardinal assumptions answer many questions left open by ZFC in the desired way, the question whether they are true and should therefore be added to the standard axiomatization of set theory remains open.

Examples

- A cardinal κ is *supercompact* if for every cardinal $\lambda \geq \kappa$, there exists a normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$.
- $0^\#$ exists if and only if for some L_δ, there exists an uncountable set of indiscernibles.
- *Vopěnka's Principle* is the scheme of axioms stating that for every proper class of graphs, there are two members of the class with a homomorphism between them.

Structural reflection

In order to address the problems discussed above, Bagaria introduced a framework of canonical strengthenings of the Downward Löwenheim-Skolem Theorem that aims to include various large cardinal assumptions.

This framework is based on the following type of reflection principles:

Definition (Bagaria)

Given a class $\mathcal{C}$ of structures[1] of the same type and an infinite cardinal κ, we let $\mathrm{SR}_{\mathcal{C}}(\kappa)$ denote the statement that for every structure B in $\mathcal{C}$, there exists a structure A in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of A into B.

[1] In the following, the term *structure* refers to structures for countable first-order languages.

Note that the Downward Löwenheim-Skolem Theorem implies that the principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every elementary class $\mathcal{C}$ of structures and every uncountable cardinal κ.

We can extend this result by considering classes of structures defined by formulas of low set-theoretic complexity.

Definition

- A formula in the language $\mathcal{L}_{\in}$ of set theory is a Σ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_{\in}$-formulas that contains all atomic $\mathcal{L}_{\in}$-formulas and is closed under negation, disjunction and bounded quantification.

- An $\mathcal{L}_{\in}$-formula is a Σ_{n+1}-*formula* if it is of the form $\exists x \ \neg\varphi(x)$ for some Σ_n-formula φ.

Proposition

$\mathrm{SR}_{\mathcal{C}}(\kappa)$ *holds for every uncountable cardinal* κ *and every class* $\mathcal{C}$ *of structures of the same type that is definable by a* Σ_1*-formula with parameters in* $\mathrm{H}(\kappa)$*.*

Proof.

Fix a Σ_1-formula $\varphi(v_0, v_1)$ and $z \in \mathrm{H}(\kappa)$ with $\mathcal{C} = \{A \mid \varphi(A, z)\}$.

Pick $B \in \mathcal{C}$, a cardinal θ with $B \in \mathrm{H}(\theta)$ and an elementary submodel X of $\mathrm{H}(\theta)$ of cardinality less than κ with $\mathrm{tc}(\{z\}) \cup \{B\} \subseteq X$.

Let $\pi : X \longrightarrow M$ denote the corresponding transitive collapse and set $A = \pi(B)$. Then $\pi(z) = z$ and the fact that $\varphi(B, z)$ holds in $\mathrm{H}(\theta)$ implies that $\varphi(A, z)$ holds in both M and V.

This shows that A is a structure in $\mathcal{C}$ and, since $A \subseteq M$, it follows that $\pi \restriction A : A \longrightarrow B$ is an elementary embedding. $\square$

We now show how the large cardinal axioms listed above can be characterized through the principle SR.

Theorem (Bagaria et al.)

The following statements are equivalent for every cardinal κ:

- *The cardinal κ is the least supercompact cardinal.*
- *The cardinal κ is the least cardinal with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Theorem (Bagaria et al.)

The following schemes are equivalent over $\mathbf{ZFC}$*:*

- *Vopěnka's Principle.*
- *For every class* $\mathcal{C}$ *of structures of the same type, there exists a cardinal* κ *with* $\mathrm{SR}_{\mathcal{C}}(\kappa)$*.*

Theorem (Bagaria)

The following statements are equivalent:

- $0^{\#}$ *exists.*
- *For every class* $\mathcal{C}$ *of constructible structures of the same type that is definable in* L*, there exists a cardinal* κ *with* $\mathrm{SR}_{\mathcal{C}}(\kappa)$*.*

Supercompact cardinals

The following result extends the connection between supercompactness and the validity of the principle SR for Σ_2-definable classes.

Theorem (Bagaria)

The following statements are equivalent for every cardinal κ:

- *The principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
- *The cardinal κ is either supercompact or a limit of supercompact cardinals.*

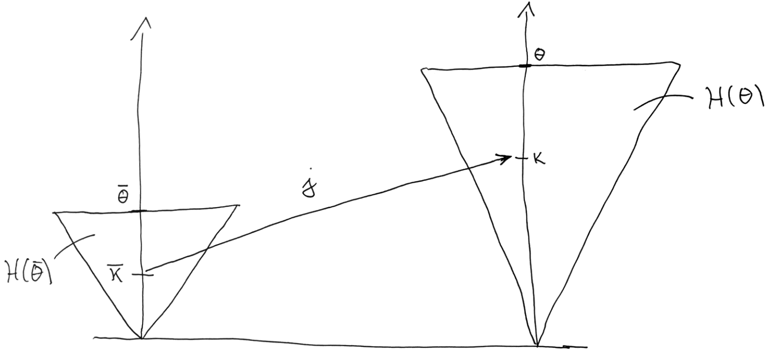
We sketch how the above equivalence can be derived from classical results of Magidor.

Lemma (Magidor)

The following statements are equivalent for every cardinal κ:

- κ *is a supercompact cardinal.*
- *For all cardinals* $\theta > \kappa$ *and all* $z \in \mathrm{H}(\theta)$*, there exist*
    - *cardinals* $\bar{\kappa} < \bar{\theta} < \kappa$*, and*
    - *an elementary embedding* $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$

  such that $\bar{\kappa} = \mathrm{crit}(j)$, $j(\bar{\kappa}) = \kappa$ *and* $z \in \mathrm{ran}(j)$.



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K
j
θ̄
H(θ̄)
K̄

Lemma

If κ is a supercompact cardinal, then $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.

Proof.

Pick a Σ_2-formula $\varphi(v_0, v_1)$ and $z \in \mathrm{V}_\kappa$ with $\mathcal{C} = \{A \mid \varphi(A, z)\}$.

Fix $B \in \mathcal{C}$. Then $\varphi(B, z)$ holds and there exists a cardinal $\theta > \kappa$ such that $B \in \mathrm{H}(\theta)$ and $\varphi(B, z)$ holds in $\mathrm{H}(\theta)$.

Then there exists a cardinal $\bar{\theta} < \kappa$ and a non-trivial elementary embedding $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$ with $j(\mathrm{crit}(j)) = \kappa$ and $B, z \in \mathrm{ran}(j)$.

Pick $A \in \mathrm{H}(\bar{\theta})$ with $j(A) = B$. Then $z \in \mathrm{H}(\bar{\theta})$ with $j(z) = z$ and $\varphi(A, z)$ holds in $\mathrm{H}(\bar{\theta})$ and V.

This allows us to conclude that $A \in \mathcal{C}$ and $j \restriction A : A \longrightarrow B$ is an elementary embedding. $\square$

Lemma (Magidor)

Given limit ordinals $\alpha < \beta$, if there exists a non-trivial elementary embedding $j : \mathrm{V}_\alpha \longrightarrow \mathrm{V}_\beta$, then $\mathrm{crit}(j)$ is μ-supercompact for all $\mathrm{crit}(j) \leq \mu < \alpha$.

Lemma

Let κ be a cardinal with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ. If κ is not a supercompact cardinal, then κ is a limit of supercompact cardinals.

Proof.

Fix $\alpha < \kappa$. Since κ is not supercompact, there exists a cardinal $\theta > \kappa$ with the property that for every cardinal $\bar{\theta} < \kappa$, there is no non-trivial elementary embedding $j : \mathrm{H}(\bar{\theta}) \longrightarrow \mathrm{H}(\theta)$ with $j(\mathrm{crit}(j)) = \kappa$.

Note that our assumption directly implies that $\mathrm{H}(\kappa) = \mathrm{V}_\kappa \prec_{\Sigma_2} \mathrm{V}$.

Pick a cardinal $\vartheta > \theta$ such that $\mathrm{H}(\vartheta)$ is sufficiently elementary in V. Then our assumption yields cardinals $\alpha < \bar{\kappa} < \bar{\theta} < \bar{\vartheta} < \kappa$ with $\mathrm{H}(\bar{\kappa}) = \mathrm{V}_{\bar{\kappa}}$ and $\mathrm{H}(\bar{\vartheta}) = \mathrm{V}_{\bar{\vartheta}}$ and an elementary embedding $j : \mathrm{H}(\bar{\vartheta}) \longrightarrow \mathrm{H}(\vartheta)$ with $j(\alpha) = \alpha$, $j(\bar{\kappa}) = \kappa$, $j(\bar{\theta}) = \theta$ and $j(\bar{\vartheta}) = \vartheta$.

We then know that $\alpha < \mathrm{crit}(j) < \bar{\kappa}$ and hence $\mathrm{crit}(j)$ is μ-supercompact for all $\mathrm{crit}(j) \leq \mu < \bar{\kappa}$.

By elementarity, we know that $j(\mathrm{crit}(j))$ is μ-supercompact for all $j(\mathrm{crit}(j)) \leq \mu < \kappa$ and the fact that $\mathrm{V}_\kappa \prec_{\Sigma_2} \mathrm{V}$ then implies that $j(\mathrm{crit}(j))$ is supercompact. □

Strongly unfoldable cardinals

We know aim to show that the above connection between large cardinal properties and structural reflection can be generalized to other regions of the large cardinal hierarchy.

More specifically, we will show that a natural weakening of supercompactness can be canonically characterized through a weakening of the above reflection principle.

The following large cardinal property was introduced by Villaveces in his investigation of chains of end elementary extensions of models of set theory.

Definition (Villaveces)

An inaccessible cardinal κ is *strongly unfoldable* if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $V_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) \geq \lambda$.

The next result shows why strong unfoldability cardinals can be seen as a miniature version of supercompactness.

Lemma (Džamonja–Hamkins)

An inaccessible cardinal κ is strongly unfoldable if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with ${}^{\lambda}N \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) \geq \lambda$.

Strongly unfoldable cardinals turn out to have a very rich structure theory. In particular, many important results about supercompact cardinals have analogs for strongly unfoldable cardinals.

Theorem (Hamkins–Johnstone)

The following statements are equiconsistent over $\mathbf{ZFC}$*:*

- *There exists a strongly unfoldable cardinal.*
- *The restriction of the Proper Forcing Axiom to the class of proper partial orders that preserve either* $\aleph_2$ *or* $\aleph_3$*.*

Theorem (Džamonja–Hamkins)

If the existence of a strongly unfoldable cardinal is consistent with the axioms of $\mathbf{ZFC}$*, then a failure of the principle* $\diamondsuit_{Reg}(\kappa)$ *at a strongly unfoldable cardinal* κ *is consistent with these axioms.*

Theorem

If κ *is a strongly unfoldable cardinal with* $\mathcal{P}(\kappa) \subseteq \mathrm{HOD}_z$ *for some* $z \subseteq \kappa$*, then* $\diamondsuit_{Reg}(\kappa)$ *holds.*

The following large cardinal property, introduced by Rathjen in a proof-theoretic context, will be central for our further analysis:

Definition (Rathjen)

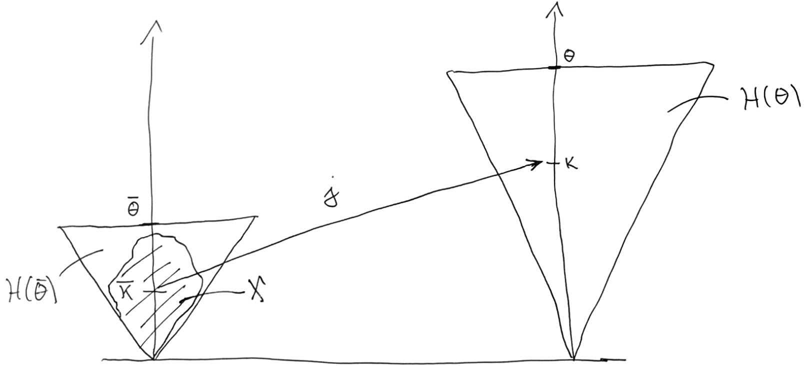
A cardinal κ is *shrewd* if for every $\mathcal{L}_\in$-formula $\Phi(v_0, v_1)$, every $\gamma > \kappa$ and every $A \subseteq \mathrm{V}_\kappa$ such that $\Phi(A, \kappa)$ holds in V_γ, there exist $\alpha < \beta < \kappa$ such that $\Phi(A \cap \mathrm{V}_\alpha, \alpha)$ holds in V_β.

Lemma

The following statements are equivalent for every cardinal κ:

- *κ is a shrewd cardinal.*
- *For all cardinals $\theta > \kappa$ and all $z \in \mathrm{H}(\theta)$, there exist*
  - *cardinals $\bar{\kappa} < \bar{\theta} < \kappa$,*
  - *an elementary submodel X of $\mathrm{H}(\bar{\theta})$, and*
  - *an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$*

  such that $\bar{\kappa} + 1 \subseteq X$, $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.


θ
H(θ)
κ
j
θ̄
H(θ̄)
κ̄
X

It turns out that the above indescribability property is equivalent to strong unfoldability.

Theorem

A cardinal is strongly unfoldable if and only if it is shrewd.

We now use these characterizations of strong unfoldability to obtain another characterization based on principles of structural reflection.

Definition (Bagaria–Väänänen)

Let $\mathcal{C}$ be a non-empty class of structures of the same type and let κ be an infinite cardinal.

- $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ denotes the statement that for every structure B in $\mathcal{C}$ of cardinality κ, there exists a structure A in $\mathcal{C}$ of cardinality less than κ and an elementary embedding of A into B.
- $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ denotes the statement that $\mathcal{C}$ contains a structure of cardinality less than κ.

The principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ obviously implies both $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$.

It can be shown that no other implication between these principles holds in general.

Theorem

The following statements are equivalent for every cardinal κ:

- *The principles $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
- *The cardinal κ is either strongly unfoldable or a limit of supercompact cardinals.*

Corollary

The following statements are equivalent for every cardinal κ:

- *The cardinal κ is the least strongly unfoldable cardinal.*
- *The cardinal κ is the least cardinal with the property that the principles $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Corollary

The following statements are equivalent for every singular cardinal κ:

- *The principle $\mathrm{SR}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*
- *The principles $\mathrm{SR}^{-}_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ.*

Observation

Given a cardinal κ, the principle $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ holds for every class $\mathcal{C}$ of structures of the same type that is definable by a Σ_2-formula with parameters in V_κ if and only if $\mathrm{V}_\kappa \prec_{\Sigma_2} \mathrm{V}$.

Observation

Let κ be an inaccessible cardinal that is not shrewd. Let $\Phi(v_0, v_1)$, $\gamma > \kappa$ and $A \subseteq \mathrm{V}_\kappa$ witness this. Assume that there are

- infinite cardinals $\rho < \bar{\kappa} < \bar{\theta} < \kappa < \theta$,
- an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\mathrm{V}_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j(\rho) = \rho$, $j(\bar{\kappa}) = \kappa$ and $A, \gamma \in \mathrm{ran}(j)$.

Then $j \restriction \mathrm{V}_{\bar{\kappa}} : \mathrm{V}_{\bar{\kappa}} \longrightarrow \mathrm{V}_\kappa$ is a non-trivial elementary embedding with $\mathrm{crit}(j) > \rho$.

Vopěnka cardinals

Above, we showed how the validity of the first-order Vopěnka's Principle can be characterized through principles of structural reflection.

We now want to characterize the validity of the second-order Vopěnka's Principle in initial segments of the set-theoretic universe.

Definition

An inaccessible cardinal δ is a *Vopěnka cardinal* if for every set $\mathcal{C} \in \mathrm{V}_{\delta+1} \setminus \mathrm{V}_\delta$ of graphs, there are two members of the class with a homomorphism between them.

Theorem

The following statements are equivalent for every uncountable cardinal δ*:*

- *For every set* $\mathcal{C}$ *of structures of the same type with* $\mathcal{C} \subseteq V_\delta$*, there exists a cardinal* $\kappa < \delta$ *with the property that the principle* $\mathrm{SR}_{\mathcal{C}}(\kappa)$ *holds.*
- *The cardinal* δ *is a Vopěnka cardinal.*

The main challenge in the proof of the above equivalence is to show that the given reflection property implies inaccessibility. The following observation presents the main idea of the proof of this implication.

Observation

If δ is a singular cardinal of countable cofinality, then there is a set $\mathcal{C} \subseteq \mathrm{V}_\delta$ of groups with the property that $\mathrm{SR}_{\mathcal{C}}(\kappa)$ fails for every cardinal $\kappa < \delta$.

Proof.

Let $\langle \kappa_n \mid n < \omega \rangle$ denote a strictly increasing sequence of infinite cardinals that is cofinal δ.

Given $1 < n < \omega$, let G_n denote the sum of κ_n-many copies of $\mathbb{Z}/n\mathbb{Z}$.

Define $\mathcal{C} = \{G_n \mid 1 < n < \omega\} \subseteq \mathrm{V}_\delta$.

Given a cardinal $\kappa < \delta$ and a prime p with $\kappa_p > \kappa$, there is no elementary embedding of a group of cardinality less than κ in $\mathcal{C}$ into G_p. $\square$

Subtle cardinals

We now show that, by replacing all occurrences of the principle SR in the above characterization of Vopěnka cardinals with the conjunction of the principles SR^{-} and SR^{--}, we obtain a characterization of another well-known large cardinal notion from the lower part of the large cardinal hierarchy.

Remember that, given a set A of ordinals, a sequence $\langle E_\alpha \mid \alpha \in A\rangle$ is an A-*list* if $E_\alpha \subseteq \alpha$ holds for every $\alpha \in A$.

Definition (Jensen–Kunen)

An infinite cardinal δ is *subtle* if for every δ-list $\langle E_\gamma \mid \gamma < \delta\rangle$ and every closed unbounded subset C of δ, there exist $\beta < \gamma$ in C with $E_\beta = E_\gamma \cap \beta$.

Lemma

If δ is a subtle cardinal, then there are stationary-many $\kappa < \delta$ with the property that κ is strongly unfoldable in V_δ.

Theorem (Jensen–Kunen)

If δ is a subtle cardinal, then $\diamondsuit_{Reg}(\delta)$ holds.

Theorem

The following statements are equivalent for every uncountable cardinal δ:

- *For every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principles $\mathrm{SR}^-_\mathcal{C}(\kappa)$ and $\mathrm{SR}^{--}_\mathcal{C}(\kappa)$ hold.*
- *The cardinal δ is either subtle or a limit of subtle cardinals.*

Corollary

The following statements are equivalent for every uncountable cardinal δ:

- *The cardinal δ is the least subtle cardinal.*
- *The cardinal δ is the least cardinal with the property that for every set $\mathcal{C}$ of structures of the same type with $\mathcal{C} \subseteq V_\delta$, there exists a cardinal $\kappa < \delta$ with the property that the principles $\mathrm{SR}^-_{\mathcal{C}}(\kappa)$ and $\mathrm{SR}^{--}_{\mathcal{C}}(\kappa)$ hold.*

$$\frac{\textit{Vopěnka}}{\textit{supercompact}} = \frac{\textit{subtle}}{\textit{strongly unfoldable}} = \frac{\textit{Woodin}}{\textit{strong}}$$

Thank you for listening!