Some Ramsey theory and topological dynamics for first order theories

Krzysztof Krupiński
(joint work with Junguk Lee and Slavko Moconja)

Instytut Matematyczny
Uniwersytet Wrocławski

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General look on the subject

The aim of Kechris, Pestov, Todorčević (or KPT) theory is to study relationships between combinatorial (Ramsey theoretic) properties of a Fraïssé class and dynamical properties of the automorphism group of the Fraïssé limit, and to find mutual applications. A general motivation for the research that I am going to discuss is to describe relationships between "definable" Ramsey theoretic and dynamical properties of first order theories. On the one hand, this leads to counterparts of some results from KPT theory; on the other, to a new kind of results. One of the key properties that we want to understand is when the Ellis group of the given theory is [pro]finite or even trivial. We find it interesting in its own right, but also, using the continuous epimorphism from the Ellis group to $\mathrm{Gal}_{KP}(T)$ (i.e. Kim-Pillay Galois group of T), profiniteness implies that $\mathrm{Gal}_{KP}(T)$ is profinite, equivalently $E_{KP} = E_{Sh}$ (i.e. Kim-Pillay strong types coincide with Shelah strong types) which is a fundamental property.

ERP for a structure

Let M be any structure. For a finite tuple $\bar{a}$ from M and $C \subseteq M$:

$$\binom{C}{\bar{a}} := \{\bar{a}' \subseteq C : \bar{a}' = f(\bar{a}) \text{ for some } f \in \mathrm{Aut}(M)\}.$$

Definition

A structure M has ERP (the *embedding Ramsey property*) if for any finite tuple $\bar{a}$ from M and a finite set $B \subseteq M$ containing $\bar{a}$, for any $r \in \omega$, for every coloring $c \colon \binom{M}{\bar{a}} \to r$, there is $B' \in \binom{M}{B}$ such that $\binom{B'}{\bar{a}}$ is monochromatic with respect to c (where $\binom{M}{B}$ is defined analogously to $\binom{C}{\bar{a}}$) .

Examples

Many Fraïssé structures: $(\mathbb{Q}, \leq)$, ordered random graph or hypergraph, Fraïssé limit of the class of all finite n linear orders, ...

Ramsey theoretic characterization of extreme amenability

Definition

Let G be a topological group. A G*-flow* is a pair (G, X), where $X \neq \emptyset$ is compact and G acts continuously on X.

Example: Bernoulli shift

$G := \mathbb{Z}$ acts on $X := 2^{\mathbb{Z}}$ via $(k * f)(n) := f(n-k)$.

Definition

A topological group G is *extremely amenable* if every G-flow (G, X) has a fixed point.

Theorem (a generalization of a KPT theorem)

Let M be any structure. Then M has the ERP iff $\mathrm{Aut}(M)$ is extremely amenable (as a topological group equipped with the pointwise convergence topology).

Extremely amenable theories

Let T be a (complete) first order theory, and $\mathfrak{C} \models T$ a monster (or just $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous) model. For $p \in S_{\bar{x}}(\emptyset)$, $S_p(\mathfrak{C}) := \{q \in S_{\bar{x}}(\mathfrak{C}) : p \subseteq q\}$. Then $\mathrm{Aut}(\mathfrak{C})$ acts naturally on $S_p(\mathfrak{C})$ turning it into an $\mathrm{Aut}(\mathfrak{C})$-flow.

Definition (Hrushovski, Krupiński, Pillay)

T is *extremely amenable* if for every $p \in S(\emptyset)$ the flow $(\mathrm{Aut}(\mathfrak{C}), S_p(\mathfrak{C}))$ has a fixed point (i.e. an invariant type).

Fact

1. This definition does not depend on the choice of $\mathfrak{C}$.
2. If $\mathrm{Aut}(\mathfrak{C})$ is extremely amenable as a topological group, then T is extremely amenable.

Examples of extremely amenable theories

1. The theory of any countable, ω-categorical structure with extremely amenable group of automorphisms is extremely amenable, so e.g. the theory of any random ordered hypergraph.
2. The theory of any ω-categorical Fraïssé structure whose age has canonical amalgamation (e.g. free amalgamation), so e.g. any random hypergraph (while the group of automorphism of such a graph is easily seen not to be extremely amenable).
3. Any stable theory in which all types in $S(\emptyset)$ are stationary (which for example is always the case when the members of $\mathrm{acl}^{eq}(\emptyset)$ are named).

Let T be a theory, and $\mathfrak{C} \models T$ a monster model. For a finite tuple $\bar{a} \subseteq \mathfrak{C}$ and any $C \subseteq \mathfrak{C}$:

$$\binom{C}{\bar{a}} = \{\bar{a}' \in C^{|\bar{a}|} : \bar{a}' \equiv \bar{a}\}.$$

Definition (KLM)

A coloring $c : \binom{C}{\bar{a}} \to 2^n$ is *definable* if there are formulas with parameters $\varphi_i(\bar{x})$, $i < n$, such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \models \varphi_i(\bar{a}') \\ 0, & \models \neg\varphi_i(\bar{a}') \end{cases}$$

for any $\bar{a}' \in \binom{C}{\bar{a}}$ and $i < n$.

DEERP and the first correspondence

Definition (KLM)

1. T has *EERP* (the *elementary embedding Ramsey property*) if for any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $n < \omega$, and any coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c.
2. T has *DEERP* (the *definable elementary embedding Ramsey property*) if the same holds but only for *definable* colorings c.

Remark (KLM)

These definitions do not depend on the choice of the monster (or just $\aleph_0$-saturated) model $\mathfrak{C}$.

Theorem (KLM)

A theory T has DEERP iff T is extremely amenable.

More abstract topological dynamics

Let (G, X) be a flow. For every $g \in G$ we have $\pi_g \colon X \to X$ given by $\pi_g(x) := gx$.

Definition/Fact

$E(X)$ is defined as the closure of $\{\pi_g : g \in G\} \subseteq X^X$ in the pointwise convergence topology on X^X. Then $E(X)$ with $\circ$ (i.e. composition) is a left topological semigroup which is compact. It is called the *Ellis semigroup* of the flow (G, X).

Comment

Various dynamical properties of the flow (G, X) can be expressed in terms of some algebraic or topological properties of $E(X)$, but we will not go into this.

Ellis theorem

Ellis Theorem

Suppose S is a compact (Hausdorff) left topological semigroup. Then there exists a minimal left ideal $\mathcal{M}$ in S (i.e. a minimal non-empty subset for which $S\mathcal{M} \subseteq \mathcal{M}$). And every such $\mathcal{M}$ satisfies the following properties.

1. $\mathcal{M}$ is closed, and $\mathcal{M} = Ss$ for all $s \in \mathcal{M}$.
2. If $u \in J(\mathcal{M}) := \{u \in \mathcal{M} : u^2 = u\}$, then $u\mathcal{M}$ is a group with the neutral element u.
3. $\mathcal{M} = \bigsqcup_{u \in J(\mathcal{M})} u\mathcal{M}$; in particular, $J(\mathcal{M}) \neq \emptyset$.
4. For every minimal left ideal $\mathcal{N}$ of S (e.g. $\mathcal{N} = \mathcal{M}$), $u \in J(\mathcal{M})$ and $v \in J(\mathcal{N})$, we have $u\mathcal{M} \cong v\mathcal{N}$.

Ellis group of a flow

Let (G, X) be a flow. Ellis theorem applies to the Ellis semigroup $E(X)$.

Definition

The *Ellis group* of the flow (G, X) is the isomorphism type of the isomorphic groups of the form $u\mathcal{M}$ for any minimal left ideal $\mathcal{M}$ of $E(X)$ and $u \in J(\mathcal{M})$. Any group $u\mathcal{M}$ as above will also be called the Ellis group of X.

Warning

In topological dynamics, something else (but strongly related) is called the Ellis group of a flow, but in model theory we use the above terminology.

Ellis group of a theory

Let $\mathfrak{C}$ be a monster model of a theory T and $\bar{c}$ its enumeration.

$$S_{\bar{c}}(\mathfrak{C}) := \{p \in S(\mathfrak{C}) : \mathrm{tp}(\bar{c}/\emptyset) \subseteq p\}.$$

This is naturally an $\mathrm{Aut}(\mathfrak{C})$-flow.

Theorem (Krupiński, Newelski, Simon)/Definition

The Ellis group of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ does not depend on the choice of the monster (or just $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous) model $\mathfrak{C}$. We call it the *Ellis group* of T.

Comment

In topological dynamics, Ellis groups play an important role in the proofs of general structural theorems. In model theory, they allowed to explain the nature of the Lascar Galois group and strong types (Krupiński, Pillay, Rzepecki). A variant of this notion proposed by Hrushovski led him to spectacular applications to additive combinatorics (approximate subgroups).

Externally definable colorings

Definition (KLM)

A coloring $c : \binom{C}{\bar{a}} \to 2^n$ is *externally definable* if there are formulas $\varphi_0(\bar{x}, \bar{y}), \ldots, \varphi_{n-1}(\bar{x}, \bar{y})$ without parameters and types $p_0(\bar{y}), \ldots, p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$ such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y}) \\ 0, & \neg\varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y}) \end{cases}$$

for any $\bar{a}' \in \binom{C}{\bar{a}}$ and $i < n$.

Remark

A coloring $c : \binom{C}{\bar{a}} \to 2^n$ is definable iff it is externally definable via realized (in $\mathfrak{C}$) types $p_0(\bar{y}), \ldots, p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$.

Definition (KLM)

T has *EDEERP* (the *externally definable elementary embedding Ramsey property*) if for any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $n < \omega$ and any externally definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c.

Remark (KLM)

This definition does not depend on the choice of the monster (or just $\aleph_0$-saturated) model $\mathfrak{C}$.

Examples

Remark

EERP $\implies$ *EDEERP* $\implies$ *DEERP*

Remark

The theory of any $\aleph_0$-saturated Fraïssé structure with the embedding Ramsey property satisfies *EERP*. There are plenty of such examples in structural Ramsey theory.

Example (KLM)

Stable theories with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ have *EDEERP*. For example, $T := ACF_0$ with named constants from the algebraic closure of $\mathbb{Q}$ has *EDEERP*, but not *EERP*.

Example (KLM)

The theory of the random $(2, 4)$-hypergraph has *DEERP* but not *EDEERP*.

Correspondences

Let $\mathrm{Inv}_{\bar{c}}(\mathfrak{C}) := \{p \in S_{\bar{c}}(\mathfrak{C}) : p \text{ invariant}\}$.

Theorem (KLM)

T has *EDEERP* iff there exists $\eta \in E(S_{\bar{c}}(\mathfrak{C}))$ such that $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$.

Corollary (KLM)

If T has *EDEERP*, then any minimal left ideal $\mathcal{M} \lhd E(S_{\bar{c}}(\mathfrak{C}))$ is trivial, hence the Ellis group of T and $\mathrm{Gal}_L(T)$ are trivial, too.

Theorem (KLM)

TFAE

1. T has *DEERP*.
2. There is an element $\eta \in E(S_{\bar{c}}(\mathfrak{C}))$ mapping all realized types in $S_{\bar{c}}(\mathfrak{C})$ to $\mathrm{Inv}_{\bar{c}}(\mathfrak{C})$.
3. T is extremely amenable, that is $\mathrm{Inv}_{\bar{c}}(\mathfrak{C}) \neq \emptyset$.

The space $S_{\bar{c},\Delta}(\bar{p})$

Let $\Delta(\bar{x},\bar{y})$ be a finite set of formulas in variables $\bar{x},\bar{y}$, and $\bar{p}=\{p_j\}_{j<m}$ a finite set of types in $S_{\bar{y}}(\emptyset)$. For any $\bar{a}$ of length $|\bar{x}|$ (in a bigger monster model $\mathfrak{C}'$), by $\mathrm{tp}_\Delta(\bar{a}/\bar{p})$ we mean the Δ-type of $\bar{a}$ over $\bigcup_{j<m} p_j(\mathfrak{C})$.

Definition (KLM)

$S_{\bar{c},\Delta}(\bar{p}) := \{\mathrm{tp}_\Delta(\bar{c}'/\bar{p}) : \bar{c}' \subseteq \bar{\mathfrak{C}}' \text{ and } \bar{c}' \equiv \bar{c}\}$.

Remark (KLM)

$S_{\bar{c}}(\mathfrak{C}) \cong \varprojlim_{\Delta,\bar{p}} S_{\bar{c},\Delta}(\bar{p})$.

Corollary (KLM)

The Ellis group of the flow $S_{\bar{c}}(\mathfrak{C})$ is the inverse limit of the Ellis groups of the flows $S_{\bar{c},\Delta}(\bar{p})$. In particular, if the Ellis groups of the flows $S_{\bar{c},\Delta}(\bar{p})$ are finite, then the Ellis group of T is profinite.

A criterion for profiniteness of the Ellis group

Denote by $(*)$ the property of T that for all finite sets of formulas $\Delta(\bar{x}, \bar{y})$ (where $|\bar{x}| = |\bar{c}|$) and types $\bar{p} \subseteq S_{\bar{y}}(\emptyset)$, there exists $\eta \in E(S_{\bar{c},\Delta}(\bar{p}))$ with $\mathrm{Im}(\eta)$ finite.

Corollary (KLM)

$(*)$ implies that the Ellis groups of the flows $S_{\bar{c},\Delta}(\bar{p})$ are finite, so the Ellis group of T is profinite.

Example (KLM)

Stable theories satisfy $(*)$.

Our last goal is to give a combinatorial translation of $(*)$, which provides lots of examples of theories with profinite Ellis groups.

Definition (KLM)

If $\Delta(\bar{x}, \bar{y})$ is a set of formulas in variables $\bar{x}$ and $\bar{y}$ (where $|\bar{y}| = |\bar{c}|$), then an externally definable coloring c is called *externally definable Δ-coloring* if all the formulas $\varphi_i(\bar{x}, \bar{y})$'s defining c are taken from Δ and $p_0(\bar{y}), \ldots, p_{n-1}(\bar{y}) \in S_{\bar{c}}(\mathfrak{C})$.

Definition (KLM)

1. A theory T has *separately finite EERdeg* (*separately finite elementary embedding Ramsey degree*) if for any finite tuple $\bar{a}$ there exists $l < \omega$ such that for any finite tuple $\bar{b} \subseteq \mathfrak{C}$ containing $\bar{a}$, $r < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to r$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $|c[\binom{\bar{b}'}{\bar{a}}]| \leqslant l$.
2. A theory T has *separately finite EDEERdeg* (*separately finite externally definable elementary embedding Ramsey degree*) if for any finite tuple $\bar{a}$, finite set of formulas Δ in variables $\bar{x}$ (where $|\bar{x}| = |\bar{a}|$) and $\bar{y}$ (where $|\bar{y}| = |\bar{c}|$), there exists $l < \omega$ such that for any finite tuple $\bar{b} \subseteq \mathfrak{C}$ containing $\bar{a}$, $n < \omega$, and externally definable Δ-coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $|c[\binom{\bar{b}'}{\bar{a}}]| \leqslant l$.

Examples

Remark

sep. fin. *EERdeg* $\implies$ sep. fin. *EDEERdeg*

Remark

The theory of any $\aleph_0$-saturated Fraïssé structure with the finite embedding Ramsey degree has sep. fin. *EERdeg*. There are plenty of such examples in structural Ramsey theory.

Example (KLM)

Stable theories have sep. fin. *EDEERdeg*.

Example (KLM)

The theory of the random $(2, 4)$-hypergraph has sep. fin. *EERdeg* (by the above remarks) but not *EDEERP*.

The main correspondence and criterion for profiniteness of the Ellis group

Theorem (KLM)

A theory T has sep. finite *EDEERdeg* iff it has property $(*)$.

Corollary (KLM)

Every theory with sep. fin. *EDEERdeg* has profinite Ellis group.