Generalized Ultrafilters

world logic day

Bertalan Pécsi

Jan. 2026

Contents

Ultrafilters

Ultrafilters

An ultrafilter on a set X can be seen as

- a morphism $\mathbf{2}^X \to \mathbf{2}$ in the category $\mathcal{BA}$ of Boolean Algebras.

Ultrafilters

An ultrafilter on a set X can be seen as

- a morphism $\mathbf{2}^X \to \mathbf{2}$ in the category $\mathcal{BA}$ of Boolean Algebras.

where the power set is identified with the Cartesian product

$$\mathbf{2}^X = \prod_{x \in X} \mathbf{2}$$

Ultrafilters

An ultrafilter on a set X can be seen as

- a morphism $\mathbf{2}^X \to \mathbf{2}$ in the category $\mathcal{BA}$ of Boolean Algebras.

where the power set is identified with the Cartesian product

$$\mathbf{2}^X = \prod_{x \in X} \mathbf{2}$$

$$\beta(X) = \{\text{ultrafilters on } X\}.$$

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

where L^X means the categorical product $\prod_{x \in X} L$

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

where L^X means the categorical product $\prod_{\mathrm{x} \in X} L$

with canonical arrows $\pi_{\mathrm{x}} : L^X \to L$ for each $\mathrm{x} \in X$,
satisfying

- for any X-many arrows $(\alpha_{\mathrm{x}} : A \to L)_{\mathrm{x} \in X}$ there exists a unique $\alpha : A \to L^X$ which makes $\alpha_{\mathrm{x}} = \pi_{\mathrm{x}} \circ \alpha$.

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) := \{\text{generalized ultrafilters}\} = \{L^X \to L \text{ morphisms}\}$.

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) := \{\text{generalized ultrafilters}\} = \left\{L^X \to L \text{ morphisms}\right\}$.

Example:

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) := \{\text{generalized ultrafilters}\} = \left\{L^X \to L \text{ morphisms}\right\}$.

Example:

- Let $L := \mathbf{2}$ but $\mathcal{L}$ be the cat. of κ-complete Boolean algebras.

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) := \{\text{generalized ultrafilters}\} = \{L^X \to L \text{ morphisms}\}$.

Example:

- Let $L := \mathbf{2}$ but $\mathcal{L}$ be the cat. of κ-complete Boolean algebras.

Then $\gamma(X) = \{\ \kappa\text{-complete ultrafilters on } X\ \}$.

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) := \{\text{generalized ultrafilters}\} = \left\{L^X \to L \text{ morphisms}\right\}$.

Properties:

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) := \{\text{generalized ultrafilters}\} = \left\{L^X \to L \text{ morphisms}\right\}$.

Properties:

1. We get a canonical function $X \to \gamma(X)$, $\mathrm{x} \mapsto \pi_{\mathrm{x}}$.

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) := \{\text{generalized ultrafilters}\} = \left\{L^X \to L \text{ morphisms}\right\}$.

Properties:

1. We get a canonical function $X \to \gamma(X)$, $\mathrm{x} \mapsto \pi_{\mathrm{x}}$.
2. Any $f : X \to \gamma(Y)$ canonically extends to $\gamma(X) \to \gamma(Y)$.

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) \;:=\; \{\text{generalized ultrafilters}\} \;=\; \left\{L^X \to L \text{ morphisms}\right\}.$

Properties:

1. We get a canonical function $X \to \gamma(X)$, $\mathrm{x} \mapsto \pi_{\mathrm{x}}$.
2. Any $f : X \to \gamma(Y)$ canonically extends to $\gamma(X) \to \gamma(Y)$.

Consider the arrows $(f(\mathrm{x}) : L^Y \to L)_{\mathrm{x} \in X}$, they induce a unique arrow $\hat{f} : L^Y \to L^X$, which in turn induces $\circ \hat{f} : \gamma(X) \to \gamma(Y)$ that satisfies $\pi_{\mathrm{x}} \circ \hat{f} = f(\mathrm{x})$.

Generalized ultrafilters

Fix an object L of an arbitrary category $\mathcal{L}$ which has products.

Def. A *generalized ultrafilter* on a set X is

- a morphism $L^X \to L$ in the category $\mathcal{L}$.

Let $\gamma(X) := \{\text{generalized ultrafilters}\} = \left\{L^X \to L \text{ morphisms}\right\}$.

Properties:

1. We get a canonical function $X \to \gamma(X)$, $\mathrm{x} \mapsto \pi_{\mathrm{x}}$.
2. Any $f : X \to \gamma(Y)$ canonically extends to $\gamma(X) \to \gamma(Y)$.
3. A semigroup operation on a set S induces a semigroup on $\gamma(S)$.

Monads

Monads

Def. [Kleisli] A *monad* on a category $\mathcal{C}$ consists of:

- a function $T : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$
- a morphism $\eta_c : c \to T(c)$ for each $c \in \mathrm{Ob}\,\mathcal{C}$
- an operation $*$ that assigns to any $f : x \to T(y)$ an arrow $f^* : T(x) \to T(y)$.

that satisfy

Monads

Def. [Kleisli] A *monad* on a category $\mathcal{C}$ consists of:

- a function $T : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$
- a morphism $\eta_c :\ c \to T(c)$ for each $c \in \mathrm{Ob}\,\mathcal{C}$
- an operation $*$ that assigns to any $f : x \to T(y)$ an arrow $f^* : T(x) \to T(y)$.

that satisfy

1. $f^* \circ \eta_x \;=\; f$
2. ${\eta_x}^* \;=\; 1_{T(x)}$
3. $(g^* \circ f)^* \;=\; g^* \circ f^*$
   for $f : x \to T(y)$ and $g : y \to T(z)$.

Monads

Def. [Kleisli] A *monad* on a category $\mathcal{C}$ consists of:

- a function $T : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$
- a morphism $\eta_c : c \to T(c)$ for each $c \in \mathrm{Ob}\,\mathcal{C}$
- an operation $*$ that assigns to any $f : x \to T(y)$ an arrow $f^* : T(x) \to T(y)$.

that satisfy

1. $f^* \circ \eta_x = f$
2. ${\eta_x}^* \circ f = f$
3. $(g^* \circ f)^* = g^* \circ f^*$
   for $f : x \to T(y)$ and $g : y \to T(z)$.

Monads

Def. [Kleisli] A *monad* on a category $\mathcal{C}$ consists of:

- a function $T : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$
- a morphism $\eta_c : c \to T(c)$ for each $c \in \mathrm{Ob}\,\mathcal{C}$
- an operation $*$ that assigns to any $f : x \to T(y)$ an arrow $f^* : T(x) \to T(y)$.

that satisfy

1. $f^* \circ \eta_x = f$
2. ${\eta_x}^* = 1_{T(x)}$
3. $(g^* \circ f)^* = g^* \circ f^*$
   for $f : x \to T(y)$ and $g : y \to T(z)$.

Monads and generalized ultrafilters

Theorem. Every monad on $\mathcal{S}et$ is naturally isomorphic to a γ for some $\mathcal{L}$ and $L \in \mathrm{Ob}\,\mathcal{L}$.

Monads and generalized ultrafilters

Theorem. Every monad on $\mathcal{S}et$ is naturally isomorphic to a γ for some $\mathcal{L}$ and $L \in \mathrm{Ob}\,\mathcal{L}$.

Proof: Relies on the fact that every set X is $\cong \coprod_{x \in X} 1$

and uses the *Kleisli category construction* of the given monad.

Multi-valued logic

Propositional Calculus over L

Assume that $\mathcal{L}$ has *free objects*,

Propositional Calculus over L

Assume that $\mathcal{L}$ has *free objects*,
via the left adjoint $F : \mathcal{S}et \to \mathcal{L}$ of a given functor $U : \mathcal{L} \to \mathcal{S}et$

Propositional Calculus over L

Assume that $\mathcal{L}$ has *free objects*,
via the left adjoint $F : \mathcal{S}et \to \mathcal{L}$ of a given functor $U : \mathcal{L} \to \mathcal{S}et$

- Formulas with propositional variables in X:
  elements of F(X)

Propositional Calculus over L

Assume that $\mathcal{L}$ has *free objects*,
via the left adjoint $F : \mathcal{S}et \to \mathcal{L}$ of a given functor $U : \mathcal{L} \to \mathcal{S}et$

- Formulas with propositional variables in X:
  elements of $U \circ F(X)$

Propositional Calculus over L

Assume that $\mathcal{L}$ has *free objects*,
via the left adjoint $F : \mathcal{S}et \to \mathcal{L}$ of a given functor $U : \mathcal{L} \to \mathcal{S}et$

- Formulas with propositional variables in X:
  elements of F(X)
- Any evaluation $X \to L$ extends to $F(X) \to L$ arrow in $\mathcal{L}$.

Propositional Calculus over L

Assume that $\mathcal{L}$ has *free objects*,
via the left adjoint $F : \mathcal{S}et \to \mathcal{L}$ of a given functor $U : \mathcal{L} \to \mathcal{S}et$

- Formulas with propositional variables in X:
  elements of F(X)
- Any evaluation $X \to U(L)$ "extends" to an $F(X) \to L$ arrow in $\mathcal{L}$.

Propositional Calculus over L

Assume that $\mathcal{L}$ has *free objects*,
via the left adjoint $F : \mathcal{S}et \to \mathcal{L}$ of a given functor $U : \mathcal{L} \to \mathcal{S}et$

- Formulas with propositional variables in X:
  elements of F(X)
- Any evaluation $X \to L$ extends to $F(X) \to L$ arrow in $\mathcal{L}$.

Generalized Ultraproduct

An n-ary L-relation on a set A is an arrow $A^n \to L$.

Generalized Ultraproduct

An n-ary L-relation on a set A is an arrow $A^n \to L$.

We assume a binary equality(-like) L-relation $\approx$ is given with each such “first order model” A.

Generalized Ultraproduct

An n-ary L-relation on a set A is an arrow $A^n \to L$.

We assume a binary equality(-like) L-relation $\approx$ is given with each such "first order model" A.

Def. Given models $(A_i, \approx_i)$ for each $i \in I$ and an L-ultrafilter U on the index set I, the ultraproduct of A_i's is

$$\left(\prod_{i\in I} A_i\,,\ \approx\right) \quad \text{where} \quad \approx\ :=\ \mathrm{U} \circ \prod_{\mathrm{i}\in\mathrm{I}} \approx_{\mathrm{i}}$$

Generalized Ultraproduct

An n-ary L-relation on a set A is an arrow $A^n \to L$.

We assume a binary equality(-like) L-relation $\approx$ is given with each such "first order model" A.

Def. Given models $(A_i, \approx_i)$ for each $i \in I$ and an L-ultrafilter U on the index set I, the ultraproduct of A_i's is

$$\left(\prod_{i\in I} A_i\,,\ \approx\right) \quad \text{where} \quad \approx\ :=\ \mathrm{U} \circ \prod_{\mathrm{i}\in\mathrm{I}} \approx_{\mathrm{i}}\ :$$

$$\left(\prod_{i\in I} A_i\right)^2 \xrightarrow{\cong} \prod_{i\in I}(A_i)^2 \xrightarrow{\prod_i \approx_i} \prod_{i\in I} L \xrightarrow{\mathrm{U}} L\,.$$