Temporal Axioms for Minkowski spacetime

Robin Hirsch

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Abstract

Define binary relations $\leq, <, \preceq, \prec$ over $\mathbb{R}^{n+1}$ by

$$\begin{array}{lll}
(\bar{x},t) \leq (\bar{x'},t') & \iff |\bar{x}-\bar{x'}| \leq t'-t & \text{reflexive, up to sol} \\
(\bar{x},t) < (\bar{x'},t') & \iff (\bar{x},t) \leq (\bar{x'},t') \wedge (\bar{x},t) \neq (\bar{x'},t') & \text{irreflexive, up to sol} \\
(\bar{x},t) \prec (\bar{x'},t') & \iff |\bar{x}-\bar{x'}| < t'-t & \text{irreflexive, slower than light} \\
(\bar{x},t) \preceq (\bar{x'},t') & \iff (\bar{x},t) \prec (\bar{x'},t') \vee (\bar{x},t) = (\bar{x'},t') & \text{reflexive, slower than light}
\end{array}$$

The goal is to find temporal axioms, sound and complete over Minkowski spacetime $(\mathbb{R}^4, <)$ with one time dimension and three spatial dimensions. Some known results on the validity of modal/temporal formulas over various frames:

- the modal logics of $(\mathbb{R}^n, \leq)$, $(\mathbb{R}^n, \preceq)$ are axiomatised by S4.2 (reflexive, transitive, confluent) [Gol80], regardless of $n \geq 2$.
- the modal logic of $(\mathbb{R}^n, \prec)$ is axiomatised by OI.2 (transitive, serial, confluent, two-dense) [SS03], regardless of $n \geq 2$.
- the temporal logic of $(\mathbb{R}, <)$ is axiomatised by transitivity, density, linearity, Dedekind completeness [BG85].
- the validity problems of temporal formulas over $(\mathbb{R}^n, <)$, $(\mathbb{R}^n, \leq)$, $(\mathbb{R}^n, \prec)$, $(\mathbb{R}^n, \preceq)$ are EXPTIME-hard [HM22], for $n \geq 3$, but decidability is not known.
- the validity problem of temporal formulas over $(\mathbb{R}^2, \prec)$ is PSPACE-complete [Sha04, HM18], validity over $(\mathbb{R}^2, <)$ is in PSPACE [HR18][1].
- the validity problems of temporal formulas over $(\mathbb{R}^2, \leq)$, $(\mathbb{R}^2, \preceq)$ are decidable [HR18, HM18][2] and PSPACE-hard [Gol80].

To date, no axiomatisation of the temporal logic of any Minkowski spacetime frame with at least one spatial dimension is known. I will present sound and complete temporal axiomatisations of various frames, including $(\mathbb{Q}^2, <)$, and discuss possible temporal axioms for $(\mathbb{R}^2, <)$. I also raise the question of the temporal logic of certain curved spacetimes.

References

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[1] In the cited paper we also claim to prove that validity is PSPACE-hard, but there is a gap in the proof.
[2] In the cited papers we also claim to prove that validity is in PSPACE, but there is a gap in the proof.