Second-order generalized algebraic theories

Szumi Xie
Eötvös Loránd University (ELTE)
szumi@inf.elte.hu

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Examples of algebraic theories (ATs) include monoids, untyped combinator calculus, the language of formulas of propositional logic, boolean algebras. *Generalized algebraic theories* (GATs) allow multiple sorts indexed over each other, examples are: graphs, where edges are double indexed over vertices; categories, which are reflexive transitive non-simple graphs with some equations; combinatory logic (Hilbert-style proof theory for propositional logic). *Second-order algebraic theories* (SOATs) allow second-order operations, e.g. in the untyped lambda calculus, lambda is a second-order function $(\mathsf{Tm} \to \mathsf{Tm}) \to \mathsf{Tm}$. The formulas of first-order logic is a multi-sorted SOAT with separate sorts for terms and formulas, the universal quantifier is a function $(\mathsf{Tm} \to \mathsf{For}) \to \mathsf{For}$. *Second-order generalized algebraic theories* (SOGATs) merge the two features, examples are propositional logic with natural deduction–style proof theory (the sort of proofs is indexed over the sort of formulas) and first-order logic (same as propositional logic with an additional non-indexed sort for terms).

$$\begin{tikzcd}
 & \text{AT} \arrow[dl, hook] \arrow[dr, hook] & \\
\text{GAT} \arrow[dr, hook] & & \text{SOAT} \arrow[dl, hook] \\
 & \text{SOGAT} &
\end{tikzcd}$$

The second-order presentation of algebraic theories is very concise, but not well-behaved, there is no good notion of morphism between algebras (models). However we can translate second-order algebraic theories to first-order ones, which are well-behaved, e.g. the algebras form a category with free algebras, quotient algebras, initial algebras which can be seen as the syntax for the language described by the GAT, etc. The idea of the translation is that we introduce a notion of variables and we list the free variables in the context, every sort and operation become indexed by a context, second-order operations take arguments with one more free variable than what they return.

In the paper [1], we introduce signatures for GATs and SOGATs (both of which are actually SOGATs themselves) and define two different translations from SOGATs to GATs producing parallel and single substitution calculi respectively. We prove the correctness of the parallel translation with respect to a naïve presheaf-based semantics. The two translations result in different categories of models, but we conjecture that the initial algebras are the same. In the future, we would also like to describe a combinatory version of the SOGAT–GAT translation.

References

[1] Ambrus Kaposi and Szumi Xie. "Second-Order Generalised Algebraic Theories: Signatures and First-Order Semantics". In: *9th International Conference on Formal Structures for Computation and Deduction, FSCD 2024, July 10–13, 2024, Tallinn, Estonia*. Ed. by Jakob Rehof. Vol. 299. LIPIcs. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024, 10:1–10:24. DOI: 10.4230/LIPICS.FSCD.2024.10.