A completeness theorem for $L_{\infty\kappa}$

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Countable fragments of $L_{\infty\omega}$ admit a completeness theorem. In [Makkai-Reyes: First-order categorical logic, Thm. 5.1.7] this was proved by a reduction to the propositional case, where the Rasiowa-Sikorski lemma can be applied. It is known at least since [Chang: On the representation of α-complete Boolean-algebras (1957)], that the Rasiowa-Sikorski lemma can be generalized (to guarantee ultrafilters preserving κ-many given unions) at the price of assuming a distributivity property for the Boolean-algebra. In this talk I will prove a (not very closely related) variant of this, for (sufficiently small, sufficiently distributive) κ-complete lattices. These are completeness theorems for infinitary propositional logic. Just as it is much better to work with distributive lattices, rather than with sets of propositional formulas, it is better to work with "fat distributive lattices", i.e. coherent categories (or sometimes: sites/ toposes), rather than with sets of first-order formulas. I will explain this category-theoretic algebraization of first-order logic, and state the fattened-up version of our Rasiowa-Sikorski lemma, which will then give a completeness theorem for small fragments of $L_{\infty\kappa}$. The ideas are due to Christian Espíndola, the purely category-theoretic proof is joint work, and it is written down in: `https://arxiv.org/pdf/2312.12356`