DISTINGUISHING COLORINGS, PROPER COLORINGS AND COVERING PROPERTIES WITHOUT THE AXIOM OF CHOICE

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Abstract. If A is a set, the *Hartogs number* $H(A)$ is the least ordinal α such that there is no injection $f : \alpha \to A$. In 1915, Hartogs proved that $H(A)$ exists for every set A in ZF (i.e., the Zermelo–Fraenkel set theory without the Axiom of Choice (AC)).

In 1991, Galvin–Komjáth worked with cardinals in ZF to prove that the statements "Any graph has a chromatic number" and "Any graph has an irreducible proper coloring" are equivalent to AC in ZF using the above-mentioned theorem due to Hartogs.

In 1977, Babai introduced distinguishing vertex colorings under the name asymmetric colorings, and distinguishing edge colorings were introduced by Kalinowski–Pilśniak in 2015.

In 2023, Stawiski proved that the following statements are equivalent to Kőnig's Lemma (a weak form of AC) by assuming that the cardinality of color classes are well-ordered cardinals i.e., cardinals which are ordinals as well:

(a) Any infinite locally finite connected graph has a chromatic number.
(b) Any infinite locally finite connected graph has a chromatic index.
(c) Any infinite locally finite connected graph has a distinguishing number.
(d) Any infinite locally finite connected graph has a distinguishing index.

In particular, by a cardinal, Stawiski meant an ordinal that is not equinumerous with any smaller ordinal, which is a definition in ZFC.

In the absence of AC, a set m is called a cardinal if it is the cardinality $|x|$ of some set x, where $|x|$ is the set $\{y : |y| = |x| \text{ and } y \text{ is of least rank}\}$. Thus, an infinite cardinal in ZF can either be an aleph, which is an ordinal, or a set that is not well-orderable.

Results: (1). We work with simple graphs in ZF and cardinals in the absence of AC to prove that the statements (a)–(d) mentioned above are equivalent to Kőnig's Lemma (by suitably modifying a combinatorial argument of Herrlich and Tachtsis from the paper "On the number of Russell's socks or $2+2+2+\cdots=?$" published in 2006). This strengthens the above-mentioned results of Stawiski since he worked with cardinals in the presence of AC.

(2). We also formulate new conditions for the existence of irreducible proper coloring, minimal edge cover, maximal matching, and minimal dominating set in connected bipartite graphs, locally finite connected graphs, and arbitrary infinite graphs which are either equivalent to AC or Kőnig's Lemma or the principle of Dependent Choice.