Linearization of optimal rates in zero error source and channel coding problems

Nicolas Charpenay[*], Maël Le Treust[†] and Aline Roumy[‡]

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In vanishing error regime, separate encoding of independent problems is optimal. A natural illustration of this property is the *linearization* of optimal rates, which holds for both source and channel coding problems. This property does not hold in the zero error regime. Specifically, in the context of channel coding, Haemers (1979) shows a counterexample based on the Schläfli graph, where joint encoding of independent problems results in higher zero-error capacity C_0 than separate encoding. Determining when separate encoding is optimal, or equivalently when the linearization holds, remains an open problem in the zero error regime. Recently, Schrijver (2023) and (Wigderson and Zuiddam, 2023, Theorem 4.1) show that for all graphs G and G',

$$C_0(G) + C_0(G') = C_0(G \wedge G') \iff \log\left(2^{C_0(G)} + 2^{C_0(G')}\right) = C_0(G \sqcup G'), \tag{1}$$

where $\wedge$ denotes the AND product, and $\sqcup$ denotes the disjoint union of graphs. In Charpenay et al. (2023), we show that the equivalence (1) also holds for the complementary graph entropy $\bar{H}$, which according to Alon and Orlitsky (1996) and Koulgi et al. (2003), determines the optimal rate in the zero error Slepian and Wolf source coding problem.

If these two results appear similar, they differ in that for C_0 the linearization property only depends on characteristic graphs, whereas for $\bar{H}$ it may also depend on the probability distribution on the vertices of the graphs.

In this work, we establish the equivalence between *the linearization of* C_0 *and* $\bar{H}$ when the probability distribution $P_{VV'}$ maximizes a third quantity: *the zero error capacity relative to a distribution* $C(G \wedge G', P_{VV'})$. This crucial notion, introduced in Csiszár and Körner (1981), is related to C_0 via the result of (Gargano et al., 1994, Theorem 2), see also (Simonyi, 2001, Theorem 13.68), and is related to $\bar{H}$ via the result of (Marton, 1993, Lemma 1). We show that for such a distribution on the product of vertices, the linearization properties of C_0, $\bar{H}$ and $C(G, P)$ with respect to the AND product, and to the disjoint union of graphs, are all equivalent.

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[*]Univ. Rennes, CNRS, IRMAR UMR 6625, 35000 Rennes, France
[†]Univ. Rennes, CNRS, Inria, IRISA UMR 6074, 35000 Rennes, France
[‡]INRIA Rennes, Campus de Beaulieu, 35042 Rennes cedex, France