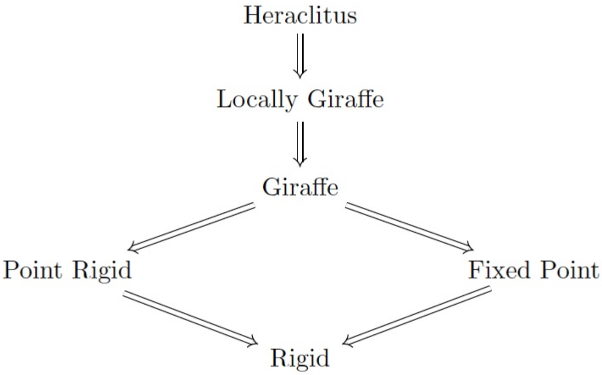
a hierarchy of spacetime symmetries: holes to heraclitus

jb manchak and thomas barrett

happy 80th istvan!


Heraclitus
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Rigid

i. isometries

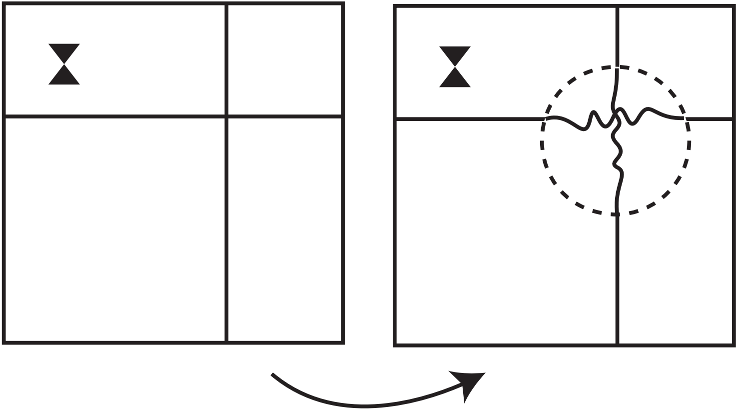
a **spacetime** is a pair (M, g) where M is a manifold and g is a lorentzian metric on M.

spacetimes (M, g) and (M', g') are **isometric** if there is a diffeomorphism $\psi : M \to M'$ such that $\psi_*(g) = g'$.

isometric spacetimes are “the same” with respect to all structure.

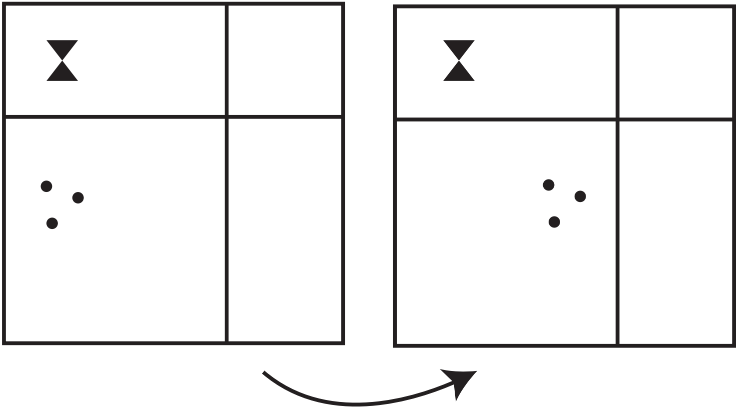
proposition. let (M, g) be any spacetime and let M' be any manifold for which there is a diffeomophism $\psi : M \to M'$. the spacetimes (M, g) and $(M', \psi_*(g))$ are isometric.

corollary. let (M, g) be any spacetime and let $\psi : M \to M$ be any diffeomorphism. the spacetimes (M, g) and $(M, \psi_*(g))$ are isometric.



isometric spacetimes (M, g) and $(M, \psi_*(g))$

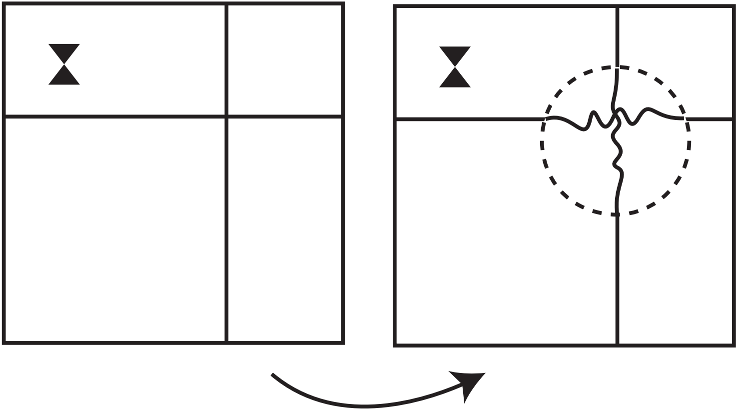
achtung! just because (M, g) and $(M, \psi_*(g))$ are isometric, it does *not* follow that ψ is an isometry of (M, g) to itself.



isometry from a spacetime to itself

ii. global symmetries

a spacetime (M, g) is **rigid** (symmetry hole-free) if for any open subset O of M and any isometry $\psi : M \to M$, if ψ is the identity on O, then ψ is the identity on all of M.



isometry from a spacetime to itself?

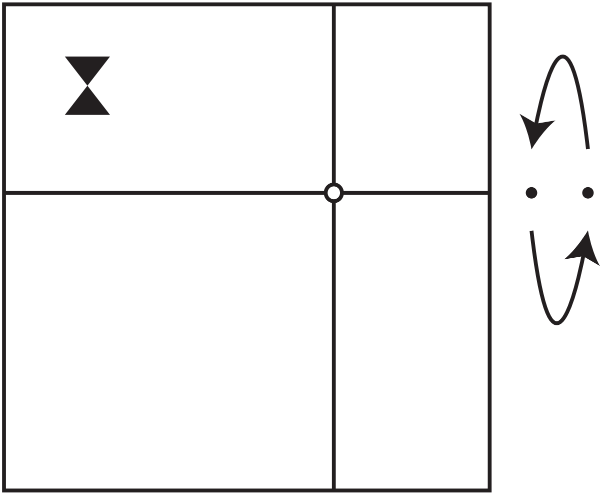
no!

proposition (halvorson and manchak 2021). any spacetime is rigid.

there are no “symmetry holes” in any spacetime.

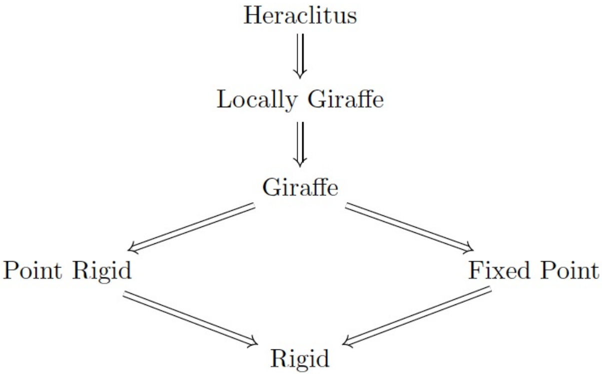
one cannot fix a spacetime outside a hole and yet “move stuff around” inside the hole.

proposition. some non-hausdorff spacetimes are not rigid.



minkowski spacetime with “two origins”

there are "symmetry holes" in some non-hausdorff spacetimes: one can fix spacetime outside a hole and yet "move stuff around" inside the hole.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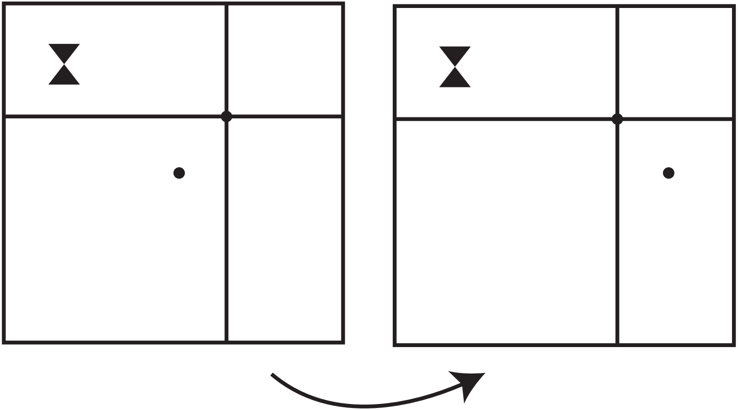
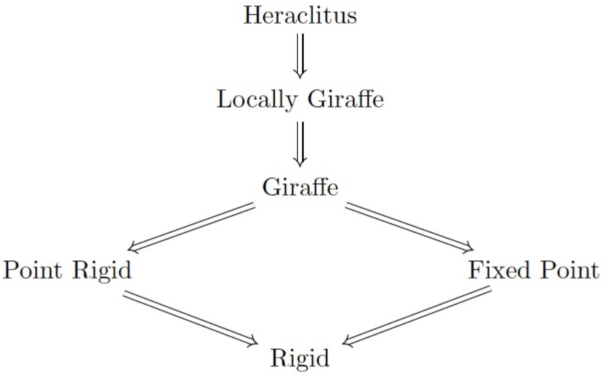
a spacetime (M, g) is **point rigid** if for any point p in M and any isometry $\psi : M \to M$, if $\psi(p) = p$, then ψ is the identity map.

proposition. any point rigid spacetime is rigid. the converse is false.



a rigid but not point rigid spacetime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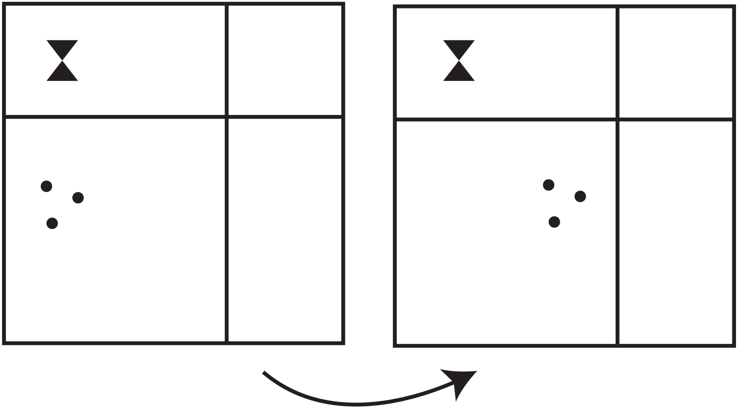
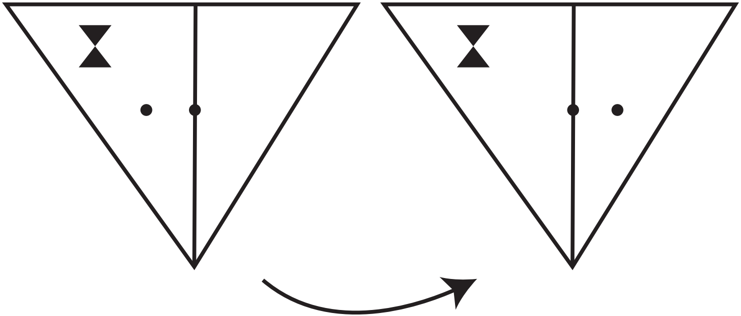
a spacetime (M, g) has a **fixed point** if there is a point p in M such that $\psi(p) = p$ for any isometry $\psi : M \to M$.

proposition. any point rigid spacetime is rigid. the converse is false.

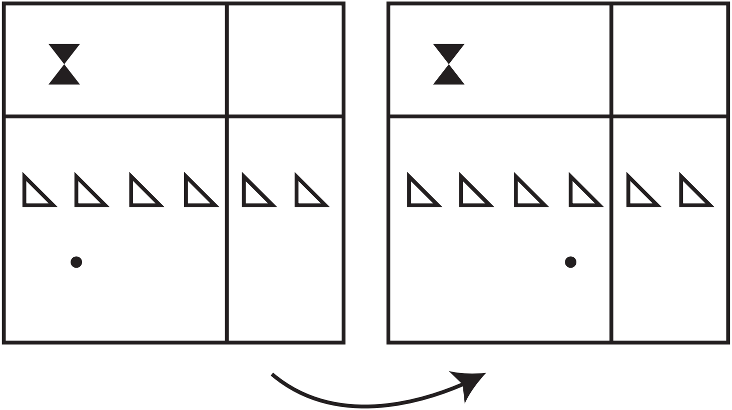


rigid spacetime without a fixed point

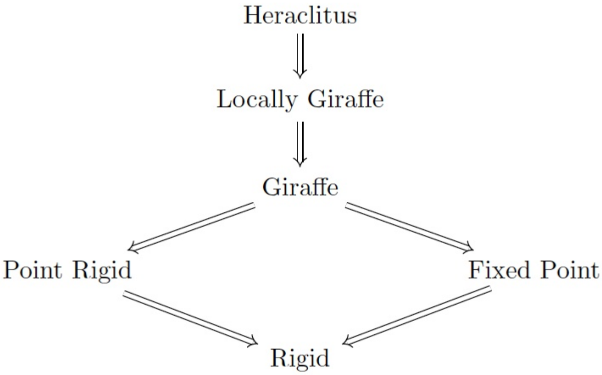
proposition. some spacetimes with a fixed point are not point rigid. some point rigid spacetimes do not have a fixed point.



spacetime with fixed point which isn't point rigid



point rigid spacetime with no fixed point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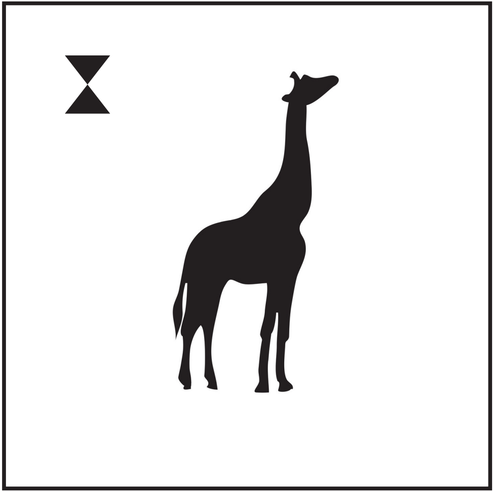
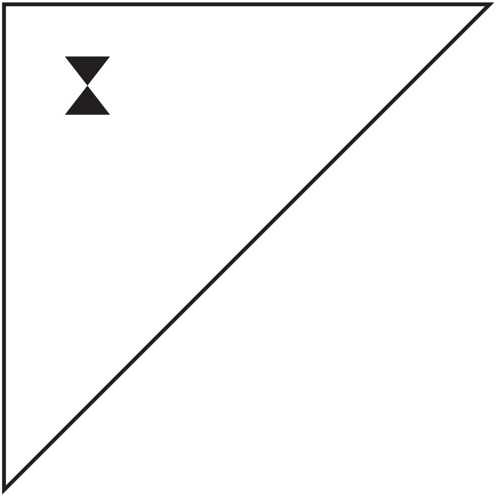
a spacetime (M, g) is **giraffe** if the identity map is the only isometry $\psi : M \to M$.

david malament: one way to construct a giraffe spacetime is to take minkowski spacetime and remove a region “shaped like a giraffe” from the manifold.

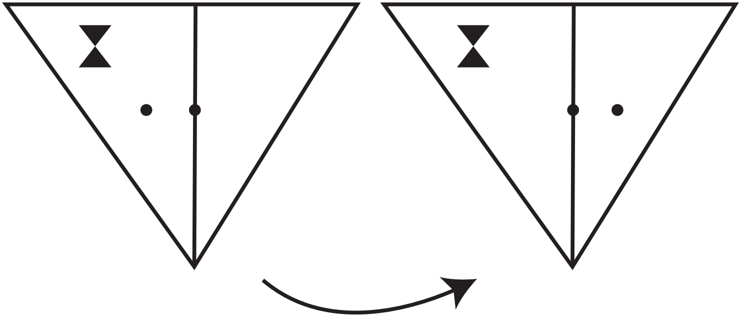


giraffe spacetime

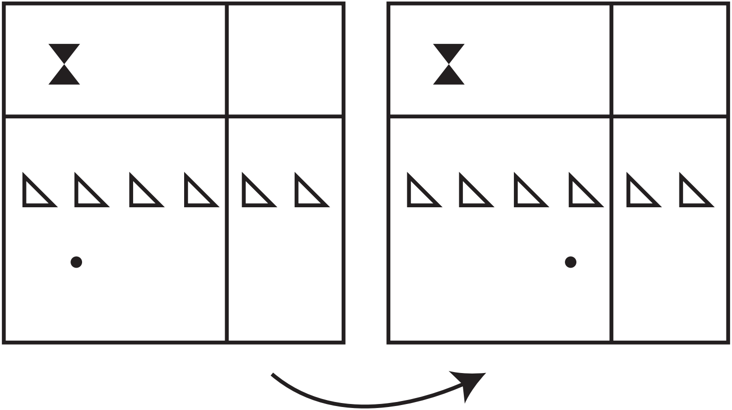


another giraffe spacetime

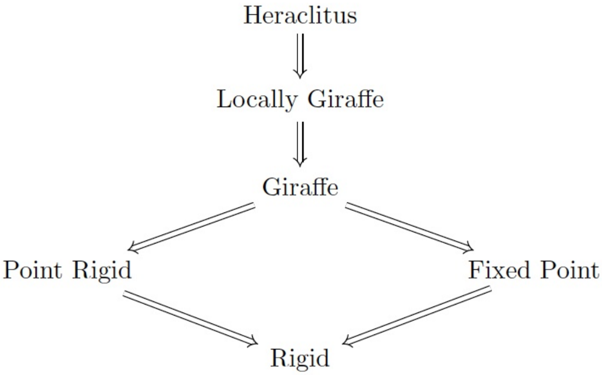
proposition. any giraffe spacetime is point rigid and has a fixed point. both converses are false.



spacetime with fixed point which isn't giraffe



point rigid spacetime which isn’t giraffe proposition. a spacetime is giraffe if and only if it is both point rigid and has a fixed point.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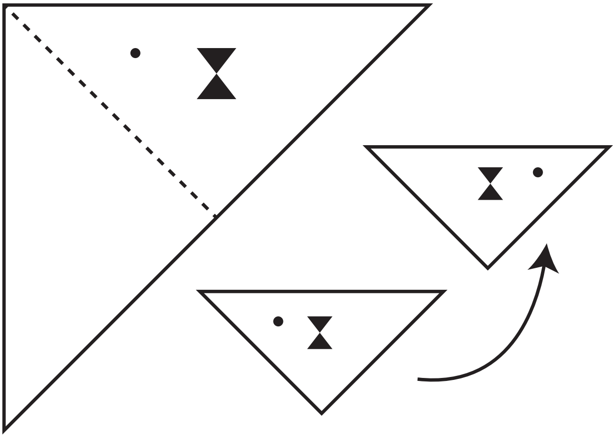
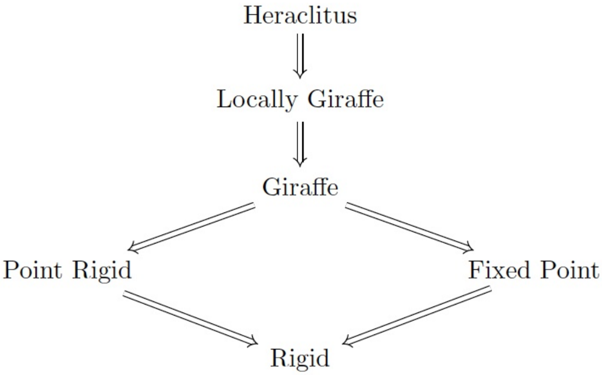
iii. local symmetries

a spacetime (M, g) is **locally giraffe** if, for any point p in M and any open connected set $O \subseteq M$, the spacetime (O, g) is giraffe.

proposition. any locally giraffe spacetime is giraffe. the converse is false.



giraffe spacetime which is not locally giraffe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a spacetime (M, g) is **heraclitus** if, for any distinct points p and q in M and any open neighborhoods O_p and O_q around these points, there is no isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$.

heraclitus spacetimes are cool.

such a spacetime is utterly devoid of (even local) symmetries.

since any neighborhoods of any distinct points fail to be isometric, each event is unlike any other.

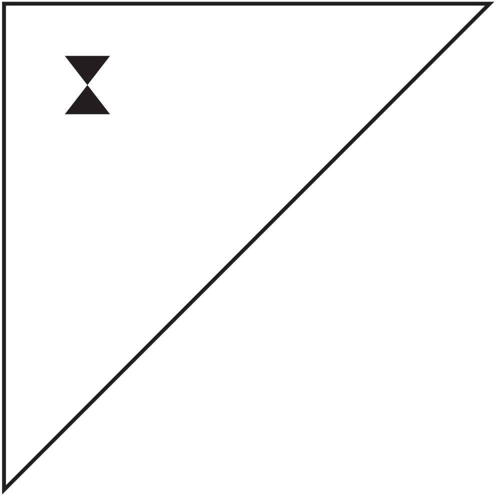
in such a spacetime, one might say that “it is impossible to step in the same river twice.”

proposition. any heraclitus spacetime is locally giraffe.

open question. is there a locally giraffe which is not heraclitus?

proposition. a heraclitus spacetime exists.

start with a portion of minkowski spacetime in standard (t, x) coordinates. call it (M, g).



(M, g) is a portion of minkowski spacetime.

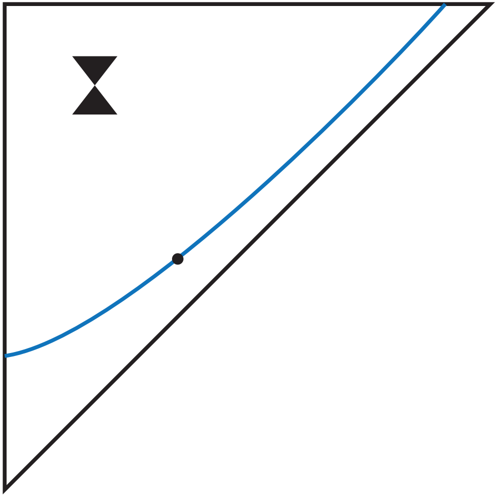
let $f : M \to \mathbb{R}$ be the euclidean distance from the origin: $f(t, x) = t^2 + x^2$.

let $\Omega : M \to \mathbb{R}$ be the function defined by $\Omega = f^{-1}$.

consider the spacetime $(M, \Omega^2 g)$. it is heraclitus.

the ricci scalar can be shown to be $R = 8(t^2 - x^2)$.

so any local isometry must map a point p to a point q with the same minkowskian distance from the origin.

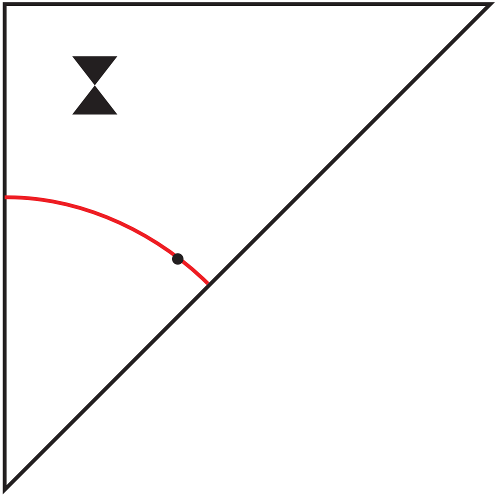


any local isometry must map a point to another with the same R value (blue line).

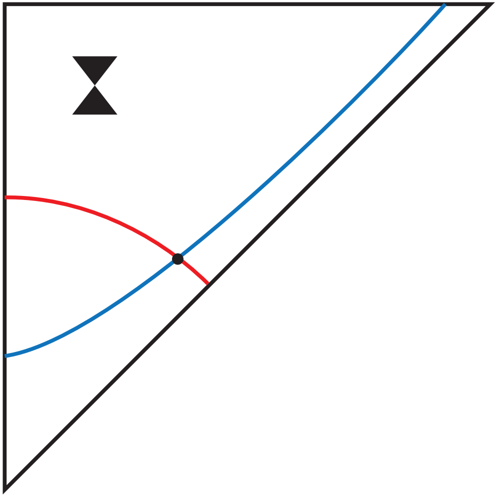
now consider the scalar $Q : M \to \mathbb{R}$ defined by $(\nabla^a R)(\nabla_a R)$.

one can show that $Q = -32Rf^2$.

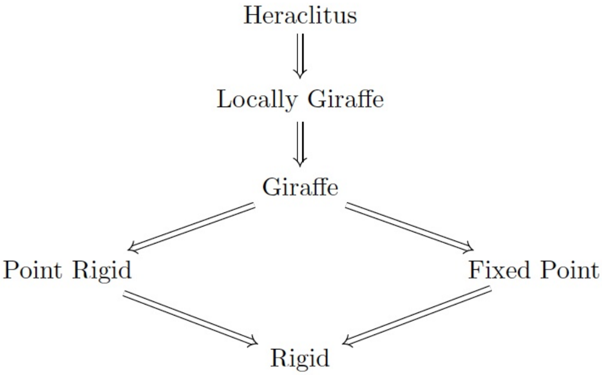
it follows that any local isometry must map a point p to a point q with the same euclidean distance from the origin.



any local isometry must map a point to another with the same f value (red line).



any local isometry must map a point to itself.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heraclitus spacetime are highly "structured" at each point allowing for some crazy uniqueness results.

spacetimes (M, g) and (M', g') are **locally isometric** if each point $p \in M$ has a neighborhood O_p that is isometric with some open set $O' \subseteq M'$ and, correspondingly, with the roles of (M, g) and (M', g') interchanged.

a property of spacetime is **local** if, given any locally isometric spacetimes, one has the property if and only if the other does.

proposition. any locally isometric heraclitus spacetimes are isometric.

corollary. given any collection of local properties, there is at most one heraclitus spacetime with exactly those properties.

thank you!