Concept Algebras of Geometries with affine reducts over ordered fields

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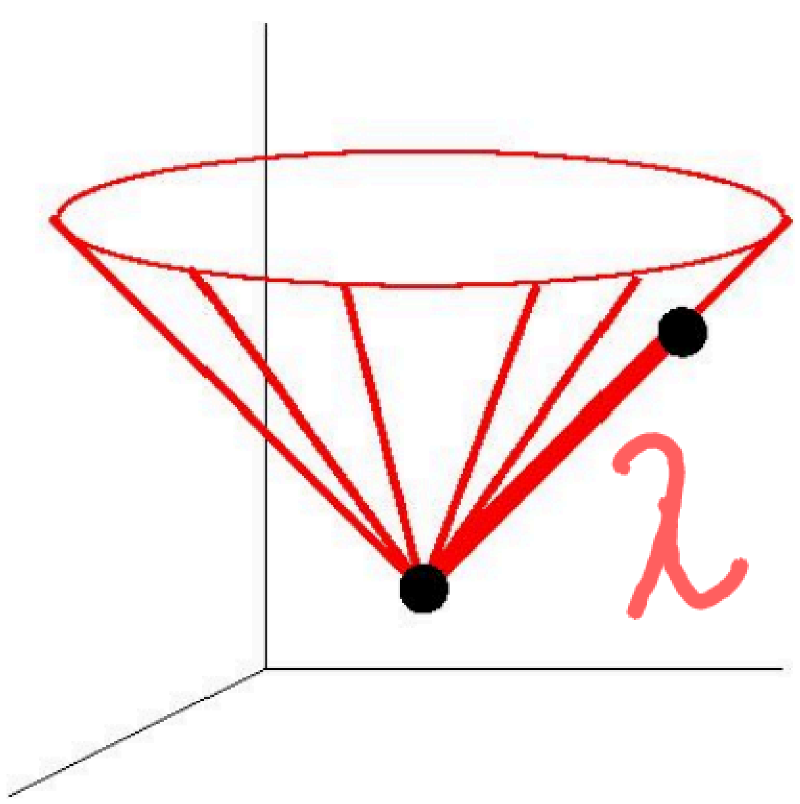
Hajnal Andréka's Conjecture

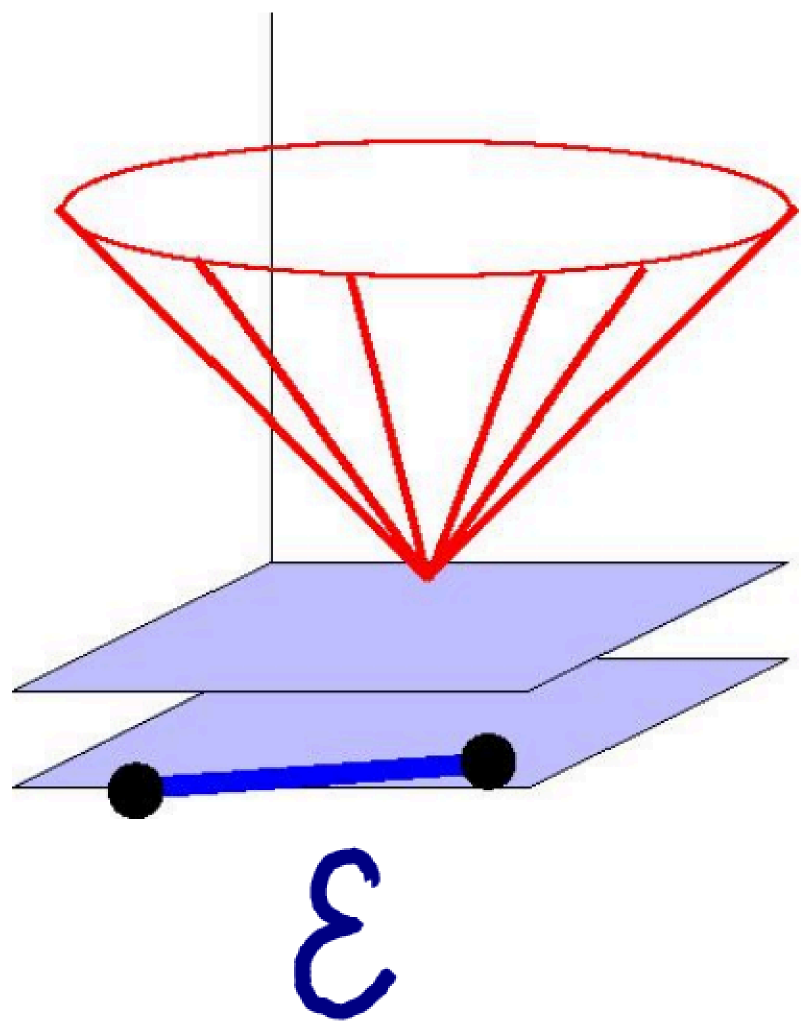
Relativistic Spacetime

Late Classical Spacetime

$\mathcal{RS} = \langle \mathbb{R}^4, \lambda \rangle$

$CS = \langle \mathbb{R}^4, \lambda, \mathcal{E} \rangle$



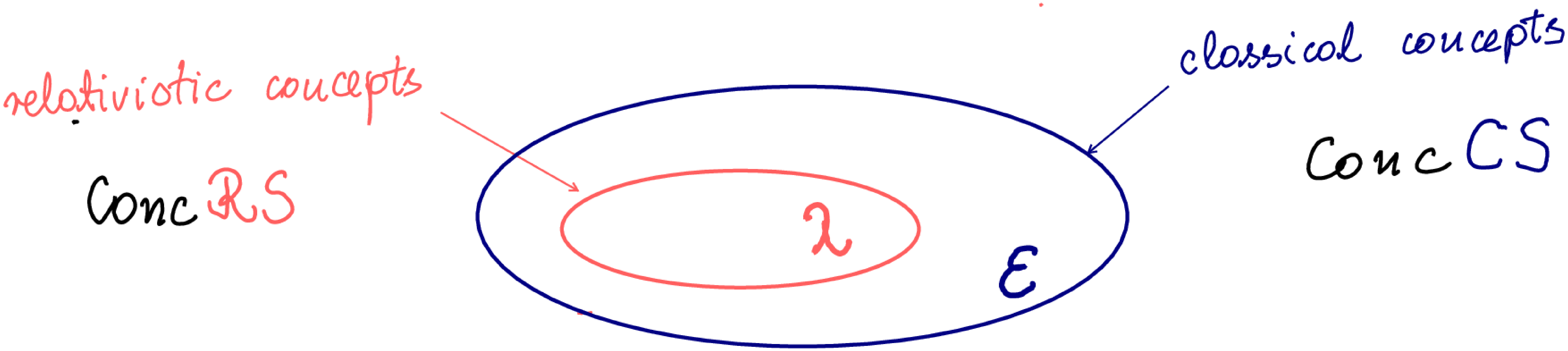


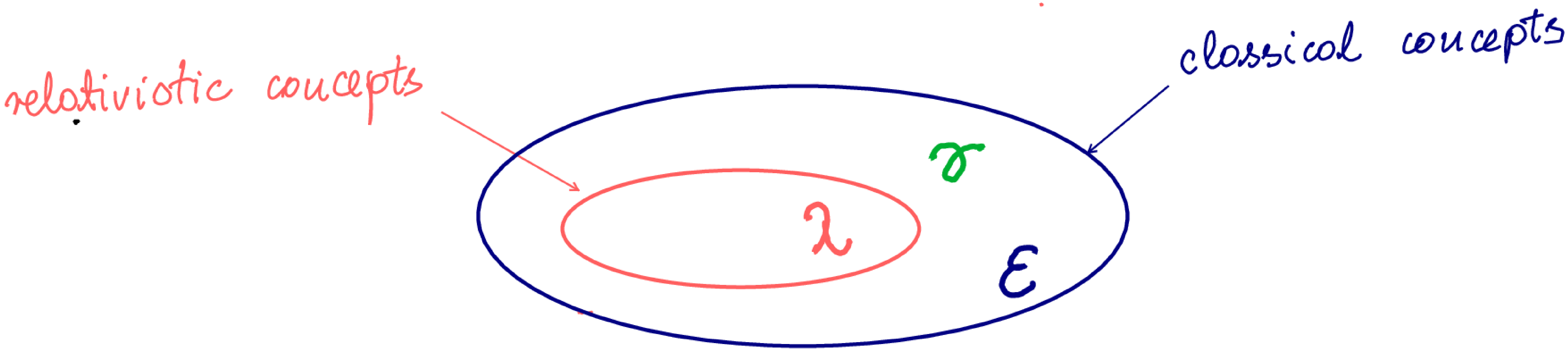
$\mathcal{M}$ FOL model

Conc $\mathcal{M}$, concepts of $\mathcal{M}$ = Definable relations

$$\mathrm{Conc}\,\mathcal{M} = \mathrm{Conc}\,\mathcal{N} \iff \mathcal{M} \equiv_{\Delta} \mathcal{N}$$ definitionally equivalent

$$\mathcal{E} \notin \mathrm{Conc}\,\mathcal{RS} = \mathrm{Conc}\,\langle \mathbb{R}^4, \lambda \rangle$$





Theorem (Hajnal's Conjecture)

$$\left.\begin{array}{l} \gamma \in \mathrm{Conc}\,\mathrm{CS} \\ \gamma \notin \mathrm{Conc}\,\mathcal{RS} \end{array}\right\} \Longrightarrow \varepsilon \in \mathrm{Conc}\langle \mathcal{RS}, \gamma \rangle$$

There is no model $\mathcal{M}$

$$\mathrm{Conc}\,\mathcal{RS} \subset \mathrm{Conc}\,\mathcal{M} \subset \mathrm{Conc}\,\mathrm{CS}$$

Concept Algebra of $\mathcal{M}$ = Cylindric Algebra of $\mathcal{M}$

$\mathfrak{Cs}\,\mathcal{M}$

Prop.

$$\mathrm{Conc}\,\mathcal{M} \subseteq \mathrm{Conc}\,\mathcal{N} \iff \mathfrak{Cs}\,\mathcal{M} \leqslant \mathfrak{Cs}\,\mathcal{N}$$

Thm. There is no concept algebra strictly between the concept algebras of $\mathcal{RS}$ and $\mathcal{CS}$.

PROOF

$d \geq 2$ dimensional geometry over ordered field $\mathcal{F} = \langle F, +, \cdot, \leq \rangle$ is $\mathcal{G} = \langle Points, \mathcal{R} \rangle$

- $\mathcal{R}$ is a finite set of relations on Points
- $Points = F^d$
- Ternary relation Bw is definable in $\mathcal{G}$
- Each $S \in \mathcal{R}$ is "definable over $\mathcal{F}$"
  field-definable

$$S \subseteq \underbrace{Points \times \ldots \times Poins}_{n} = \underbrace{F^d \times \ldots \times F^d}_{n}$$

S can be considered as $n \cdot d$-ary relation on F.

Examples

d-dimensional ordered affine geometry over $\mathcal{F}$

$\langle F^d, Bw \rangle$

Ordered affine geometries are reducts of geometries.
Therefore $\mathcal{G} = \langle F^d, \mathcal{R} \rangle$ can be coordinatised (Hilbert) and $\mathcal{F}$ "can be defined" over $\mathcal{G}$

Euclidean geometry

$\langle F^d, Bw, eq \rangle$

eq is 4-ary rel. of equidistance

Tarski's language

Proposition $\mathcal{RS} = \langle \mathbb{R}^4, \lambda \rangle$ and $\mathcal{CS} = \langle \mathbb{R}^4, \lambda, \varepsilon \rangle$ are geometries.

Proof:

$$(t,x,y,z)\ \lambda\ (t',x',y',z') \overset{\text{def}}{\Longleftrightarrow} (t-t')^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

$$(t,x,y,z)\ \varepsilon\ (t',x',y',z') \Longleftrightarrow t = t'$$

Bw is definable, moreover spacelike-, timelike-relatedness, Minkowski equidistance are definable (Goldblatt: Orthogonality and Spacetime geometry).

$\mathcal{RS}$ is the Minkowski Spacetime

$\mathrm{Aut}\,\mathcal{G}$ - automorphisms of $\mathcal{G}$

$\mathrm{AffAut}\,\mathcal{G}$ - affine automorphisms of $\mathcal{G}$

Thm. $\mathcal{G}_1$ and $\mathcal{G}_2$ are geometries over $\mathcal{F}$

$$\mathrm{Conc}\,\mathcal{G}_1 \subseteq \mathrm{Conc}\,\mathcal{G}_2$$
$$\Longleftrightarrow$$
$$\mathrm{Aut}\,\mathcal{G}_1 \supseteq \mathrm{Aut}\,\mathcal{G}_2$$
$$\Longleftrightarrow$$
$$\mathrm{AffAut}\,\mathcal{G}_1 \supseteq \mathrm{AffAut}\,\mathcal{G}_2$$

$$C_s\,\mathcal{G}_1 \leqslant C_s\,\mathcal{G}_2$$
$$\Longleftrightarrow$$
$$\mathrm{Aut}\,\mathcal{G}_1 \geqslant \mathrm{Aut}\,\mathcal{G}_2$$
$$\Longleftrightarrow$$
$$\mathrm{AffAut}\,\mathcal{G}_1 \geqslant \mathrm{AffAut}\,\mathcal{G}_2$$

Corollary $\mathbb{C}_s \mathcal{G}_1 = \mathbb{C}_s \mathcal{G}_2 \iff \mathrm{Aff\,Aut}\,\mathcal{G}_1 = \mathrm{Aff\,Aut}\,\mathcal{G}_2$

$$\mathbb{C}_s \mathcal{G}_1 < \mathbb{C}_s \mathcal{G}_2 \iff \mathrm{Aff\,Aut}\,\mathcal{G}_1 > \mathrm{Aff\,Aut}\,\mathcal{G}_2$$

There is no group between Aff Aut groups of $\mathcal{G}_1$ and $\mathcal{G}_2$

$$\Downarrow$$

There is no geometry $\mathcal{G}$ $\quad \mathbb{C}_s \mathcal{G}_1 < \mathbb{C}_s \mathcal{G} < \mathbb{C}_s \mathcal{G}_2$

$$\Updownarrow$$

There is no concept algebra $\mathbb{C}_s$ $\quad \mathbb{C}_s \mathcal{G}_1 < \mathbb{C}_s < \mathbb{C}_s \mathcal{G}_2$

Lemma

$\mathcal{S}$ is definable in $\mathcal{G}$

$\Longleftrightarrow$

$\mathcal{S}$ is $\mathcal{F}$-definable and closed under $\mathrm{Aut}\,\mathcal{G}$

$\Longleftrightarrow$

$\mathcal{S}$ is $\mathcal{F}$-definable and closed under $\mathrm{Aff}\,\mathrm{Aut}\,\mathcal{G}$.

Proof of Hajnal's Conjecture:

Aut RS = PoiSim = Poincaré $\circ$ Dilation

Aut CS = TrivSim = TrivialTr $\circ$ Dilation

Corollary of Borisov's 1978 result:

There is no group G

TrivSim < G < PoiSim

QED

Generalizations for arbitrary $d \geq 2$ and $\mathcal{F}$.

If $\mathcal{F}$ is Euclidean and $d \geq 3$ then $RS = (F^d, \lambda)$ and $CS = (F^d, \lambda, \varepsilon)$ are geometries. If $\mathcal{F}$ is not Eucl or $d = 2$, then Bw is not nec. df.

$RS = (F^d, \lambda, Col^T)$ $CS = (F^d, \lambda, col^T, \varepsilon)$

Prop: For every $\mathcal{F}$ and $d \geq 2$

AffAut RS = PoiSim AffAut CS = TrivSim

There is no G $\quad \mathrm{TrivSim} < G < \mathrm{PoiSim}$

$\Longrightarrow$ Hajnal's conjecture holds.

Thm (Gergely, Mike, Judit) For every non-Archimedean $\mathbb{F}$ and $d \geq 2$, there is a group G

$$\mathrm{TrivSim} < G < \mathrm{PoiSim}$$

Is there a geometry $\mathcal{G}$ s.t. $\mathrm{Aut}(\mathcal{G}) = G$?

Hajnal's conjecture holds for $d=4$ and real closed fields

$\Rightarrow$ For non-Archimedean real-closed fields there is no geometry g such that

$Aut(g) = G$,

$$TrivSim < G < PoiSim$$

Which subgroups of AffTr are automorphism groups of geometries?

Proposition Let $G \leq \mathrm{AffTr}$.

There is geometry $\mathcal{G}$ s.t. $\mathrm{AffAut}(\mathcal{G}) = G$

$$\Longleftrightarrow$$

G is "$\mathcal{F}$-definable"

Every element of G can be identified with (d^2+d)-tuple from F.

Then G is (d^2+d)-ary relation on F.

Hajnal's conjecture holds

$\Longleftrightarrow$

there is no "$\mathcal{F}$-definable" group $\mathrm{TrivSim} < G < \mathrm{PoiSim}$

Hajnal's conjecture fails

$\Longleftrightarrow$

there is an "$\mathcal{F}$-definable" group $\mathrm{TrivSim} < G < \mathrm{PoiSim}$

$\mathrm{TrivSim} < G < \mathrm{PoiSim}$ $\quad G$?

$V \subseteq [0,1)$

$\mathrm{PoiSim}(V) = \{ f \in \mathrm{PoiSim} : \mathrm{speed}^2(f[\text{time axis}]) \in V \}$

$\mathrm{PoiSim}(\{0\}) = \mathrm{TrivSim}$, $\mathrm{PoiSim}([0,1)) = \mathrm{PoiSim}$.

Def $V \subseteq [0,1)$ is a speed-set $\iff$ $\mathrm{PoiSim}(V)$ is a group

Trivial speed sets $\{0\}$, $[0,1)$

Thm $\mathrm{TrivSim} < G < \mathrm{PoiSim} \iff$ $G = \mathrm{PoiSim}(V)$ for some non-trivial speed set V

Prop PoiSim(V) is $\mathcal{F}$-definable $\iff$ V is definable in $\mathcal{F}$

Theorem Hajnal's conjecture holds $\iff$ there is no definable speed set,
Hajnal's conjecture fails $\iff$ there is a definable speed set

Theorem $\mathcal{F}$ is Euclidean and Archimedean and $d \geq 3$, there are no non-trivial speed sets

Theorem Hajnal's conjecture holds for every Euclidean field which is elementarily equivalent to a subfield of reals if $d \geq 3$.

Therefore in these fields there are no nontrivial definable speed-sets.

What about Euclidean non-Archimedean fields wich are not elementarily equivalent to a subfield of reals?

Thm. $\mathcal{F}$ is non-Archimedean Euclidean and $d \geq 3$.
$V \subseteq F$ is a nontrivial speed set iff

1. $V \subseteq$ infinitezimals
2. $x \in V \Rightarrow [0 \leq x \ \& \ [0,x] \subseteq V]$
3. $\exists x \ \ x > 0$
4. $x \in V \Rightarrow 2x \in V$

.

Definition V is called a cloud iff it satisfies 1–4

Theorem Let $\mathcal{F}$ be non-Archimean and Euclidean and $d \geq 3$.

Hajnal's conjecture does not hold

$\Longleftrightarrow$

there is a definable cloud in $\mathcal{F}$

QESTION Are there definable clounds?

$d \geq 3$ non-Euclidean?

Gergely: $d=2$ non-Euclideon $\Rightarrow$ there is a non triv definable speed set $\Rightarrow$ Hajnal's conjecture fails

$d=2$, field of reals: there are non-trivial speed sets $\Longrightarrow$ there are intermediate groups

there are no definable non-trivial speed sets $\Longrightarrow$ Hajnal's conjecture holds

Hajnal's Conjecture holds:

$d \geq 3$, $\mathcal{F}$ is Euclidean and elementarily equivalent to the field of reals.

$d = 2$, $\mathcal{F}$ is real-closed field

If cloud's are not definable, then Hajnal's conjecture holds for any Euclidean field, $d \geq 3$

Hajnal's Conjecture does not hold $d = 2$, $\mathcal{F}$ is non-Euclidean

Conjecture Hajnal's conjecture does not hold for any $d \geq 2$ if $\mathcal{F}$ is non-Euclidean.

$d = 2$?