# Happy Birthday!!! – LGS in Amsterdam in 1995

# Non-finitely axiomatisable canonical varieties of ‘non-relativised’ algebras of relations with infinite canonical axiomatisations

**Agi Kurucz**

*King's College London*

**Joint work with Christopher Hampson, Stanislav Kikot, and Sérgio Marcelino**

based on the paper:

C. Hampson, S. Kikot, A. Kurucz and S. Marcelino:
*Non-finitely axiomatisable modal product logics with infinite canonical axiomatisations*, **Annals of Pure and Applied Logic,** vol. 171(5):102786 (2020).

# Varieties of BAOs — normal multimodal logics

*Jónsson, Tarski, Kripke, . . .*

- **BAOs** Boolean algebras with additional operators that are
  - **normal** $\boxed{f(\dots,0,\dots) = 0}$
  - **additive** $\boxed{f(\dots,x+y,\dots) = f(\dots,x,\dots) + f(\dots,y,\dots)}$

- **normal propositional multimodal logics**
  - $\mathbf{K}$**-axioms** and **Necessitation rule** for each $\Box$ modality
  - possible world (relational aka Kripke) semantics

# Canonicity

- **canonical variety of BAOs** closed under canonical extensions

  **canonical modal logic** valid in its canonical structures

- **canonical equation** the variety it axiomatises is canonical

  **canonical formula** the modal logic it axiomatises is canonical

- *Kracht 1999*

  canonicity of an equation/formula is an **undecidable** 'semantical' property

  but: there are well-known syntactical descriptions resulting in canonical equations/formulas

  - **Sahlqvist** equations/formulas
  - **inductive** equations/formulas á la *Goranko–Vakarelov 2006*
  - . . .

# Barely canonical logics/varieties

all formulas in $\boxed{\Sigma}$ are canonical $\Longrightarrow$ the logic $\boxed{L_\Sigma}$ axiomatised by $\boxed{\Sigma}$ is canonical

there is a canonical axiomatisation for $\boxed{L}$ $\overset{?}{\Longleftarrow}$ $\boxed{L}$ is a canonical logic

- Canonicity of a logic/variety can be shown <u>without</u> finding explicit axioms:
  - *Fine 1975* **elementarily generated** logics are canonical
  - *Goldblatt 1989* logics of **ultraproduct-closed** classes are canonical
- *Hodkinson–Venema 2005*

  There are **barely canonical** logics/varieties:
  - they are canonical, but
  - every axiomatisation must contain infinitely many non-canonical axioms

FOR EXAMPLE: *Goldblatt–Hodkinson 2007, Bulian–Hodkinson 2013, Kikot 2015*

Hughes logic, McKinsey–Lemmon logic

varieties of 'non-relativised' algebras of relations: **RRA**, $\mathbf{RCA}_n$, $\mathbf{RDf}_n$ for $n \geq 3$

# Dichotomy?

but, there are many well-known **finitely Sahlqvist axiomatisable** logics/varieties

**is there anything "in between"?**

- non-finitely axiomatisable, but
- axiomatisable by (infinitely many) canonical axioms ?

Sometimes **not:**

*Kikot 2015* if $\mathcal{C}$ is FO-definable by $\forall x_0 \exists x_1 \dots \exists x_n \bigwedge x_i R_\lambda x_j$ formulas then:

- either $\mathsf{Logic\_of}(\mathcal{C})$ is barely canonical,
- or $\mathsf{Logic\_of}(\mathcal{C})$ is axiomatisable by a single inductive formula

Sometimes **yes:**

$\mathsf{Eq}(\mathsf{Crs}_n)$ **cylindric-relativised set algebras**

- *Andréka–Németi* non-finitely axiomatisable when $n \geq 3$
- *Resek–Thompson* axiomatisable by an infinite set of Sahlqvist equations

**is there any variety of 'non-relativised' algebras of relations "in between"?**

# Two-variable first-order logic with 'elsewhere' quantifiers

$$\phi := \quad P(x,y) \mid P(y,x) \mid x = y \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \exists^{\neq} x\, \phi \mid \exists^{\neq} y\, \phi$$

for some binary predicate symbols $P$

$$\begin{array}{lcll} \mathfrak{M} \models \exists^{\neq} x\, \phi[a/x, b/y] & \text{iff} & \exists a' \neq a & \mathfrak{M} \models \phi[a'/x, b/y] \\ \mathfrak{M} \models \exists^{\neq} y\, \phi[a/x, b/y] & \text{iff} & \exists b' \neq b & \mathfrak{M} \models \phi[a/x, b'/y] \end{array}$$

$$\exists x\, \phi \leftrightarrow (\phi \vee \exists^{\neq} x\, \phi) \qquad\qquad \exists y\, \phi \leftrightarrow (\phi \vee \exists^{\neq} y\, \phi)$$

The **satisfiability problem** is

- **decidable** *Grädel–Otto–Rosen 1997*
- NEXPTIME-complete *Pacholski–Szwast–Tendera 2000*
- shorter proof with connections to **integer programming** *Pratt-Hartmann 2010*

**'restricted' (equality and substitution-free) fragment:**

$$\phi := \quad P(x,y) \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \exists^{\neq} x\, \phi \mid \exists^{\neq} y\, \phi$$

# Algebraisation: ‘strict’ diagonal-free cylindric set algebras

**full rectangular set algebras:** $\boxed{\mathfrak{A} = \big(\mathcal{B}(U \times V), C_0^{\neq}, C_1^{\neq}\big)}$

for every $X \subseteq U \times V$,

$$\boxed{\begin{aligned} C_0^{\neq}(X) &= \{(u,v) : \exists u'(u' \neq u \text{ and } (u',v) \in X)\} \\ C_1^{\neq}(X) &= \{(u,v) : \exists v'(v' \neq v \text{ and } (u,v') \in X)\} \end{aligned}}$$

$$\boxed{C_i(X) = X \cup C_i^{\neq}(X)}$$

**full square set algebras:** $\boxed{\mathfrak{A} = \big(\mathcal{B}(U \times U), C_0^{\neq}, C_1^{\neq}\big)}$

- $\mathbf{sRdf_2} = \mathbb{SP}\{$full rectangular set algebras$\}$ and

  $\mathbf{sRdf_2^{\mathit{sq}}} = \mathbb{SP}\{$full square set algebras$\}$

  are (different) **discriminator** and **canonical varieties**

- $\boxed{\mathsf{Eq}(\mathbf{sRdf_2})}$ and $\boxed{\mathsf{Eq}(\mathbf{sRdf_2^{\mathit{sq}}})}$ are **decidable** $\rightsquigarrow$ **r.e.**

$\rightsquigarrow$ **let’s try to axiomatise them**

## Our results: $\mathsf{sRdf}_2$ and $\mathsf{sRdf}_2^{sq}$ are canonical varieties “in between”

$-$ $\boxed{\mathsf{Eq}(\mathsf{sRdf}_2) \quad \sim \quad \mathsf{Logic\_of}(\textit{Rectangles})}$ is **not finitely axiomatisable**

$+$ but it has an infinite **axiomatisation by Sahlqvist** equations/formulas

$-$ $\boxed{\mathsf{Eq}(\mathsf{sRdf}_2^{sq}) \quad \sim \quad \mathsf{Logic\_of}(\textit{Squares})}$ is **not finitely axiomatisable over**

$\boxed{\mathsf{Eq}(\mathsf{sRdf}_2) \quad \sim \quad \mathsf{Logic\_of}(\textit{Rectangles})}$

$+$ but it can be **axiomatised by** adding infinitely many

**Sahlqvist** equations/formulas

**Contrast:** ‘restricted’ two-variable fragment (without ‘elsewhere’ quantifiers)

- $\mathbf{Eq}(\mathbf{Rdf}_2) = \mathbf{Eq}\{\text{rectangular set algebras}\} = \mathbf{Eq}\{\text{square set algebras}\}$
  has finite Sahlqvist axiomatisation $\mathbf{Df}_2$:

  **two commuting complemented closure operators**

- $\mathbf{Eq}(\mathbf{Rdf}_2)$ is **finitely axiomatisable over** both $\mathbf{Eq}(\mathbf{sRdf}_2)$ and $\mathbf{Eq}(\mathbf{sRdf}_2^{sq})$
  just add $\boxed{x \leq c_i(x)}$

# Axiomatisation basics: ‘grids’ (of bi-clusters)

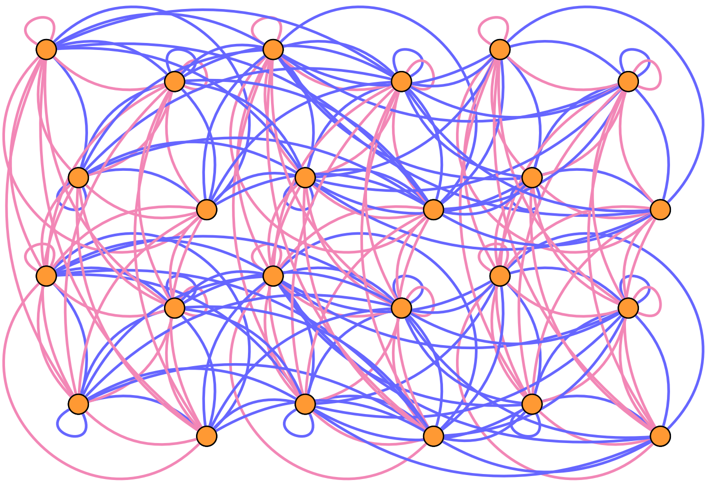
- **rectangle:** $U \times V$ with two ‘coordinate-wise $\neq$’ relations:
  $(u_1, v) \neq_0 (u_2, v)$ iff $u_1 \neq u_2$ $\qquad$ $(u, v_1) \neq_1 (u, v_2)$ iff $v_1 \neq v_2$

- Simple equationally (**Sahlqvist**) expressible properties of rectangles:

  **two commuting pseudo-equivalence relations**

  **sDf$_2$:** $x \leq -c_i(-c_i x)$
  $c_i c_i x \leq x + c_i x$
  $c_0 c_1 x = c_1 c_0 x$

- **grid:** ‘rooted’
  **sDf$_2$** atom structure



# Axiomatisation basics: 'grids' (of bi-clusters)

- **rectangle:** $U \times V$ with two 'coordinate-wise $\neq$' relations:

  $(u_1, v) \neq_0 (u_2, v)$ iff $u_1 \neq u_2$ $\qquad$ $(u, v_1) \neq_1 (u, v_2)$ iff $v_1 \neq v_2$

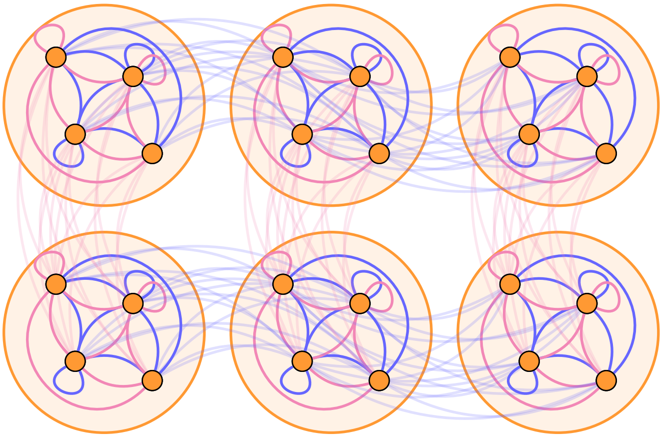
- Simple equationally (**Sahlqvist**) expressible properties of rectangles:

  **two commuting pseudo-equivalence relations**

**sDf$_2$:**
$$x \leq -c_i(-c_i x)$$
$$c_i c_i x \leq x + c_i x$$
$$c_0 c_1 x = c_1 c_0 x$$

- **grid:** 'rooted'

  **sDf$_2$** atom structure



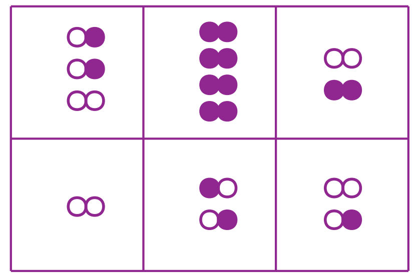
# Axiomatisation basics: ‘grids’ (of bi-clusters)

- **rectangle:** $U \times V$ with two ‘coordinate-wise $\neq$’ relations:
  $(u_1, v) \boxed{\neq_0} (u_2, v)$ iff $u_1 \neq u_2$ $\quad$ $(u, v_1) \boxed{\neq_1} (u, v_2)$ iff $v_1 \neq v_2$

- Simple equationally (**Sahlqvist**) expressible properties of rectangles:
  **two commuting pseudo-equivalence relations**

$$
\begin{aligned}
\mathbf{sDf_2}: \quad & x \leq -c_i(-c_i x) \\
& c_i c_i x \leq x + c_i x \\
& c_0 c_1 x = c_1 c_0 x
\end{aligned}
$$

- **grid:** ‘rooted’
  $\mathbf{sDf_2}$ atom structure



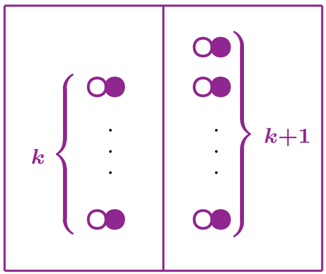
- for every finite grid $\mathfrak{F}$,
  $Cm\,\mathfrak{F} \in \mathbf{sRdf_2}$ iff $\mathfrak{F}$ is a p-morphic image of a rectangle
  $Cm\,\mathfrak{F} \in \mathbf{sRdf_2^{sq}}$ iff $\mathfrak{F}$ is a p-morphic image of a square

: $R_0$-reflexive, $R_1$-irreflexive
: $R_0$-irreflexive, $R_1$-reflexive
: both-irreflexive
: both-reflexive

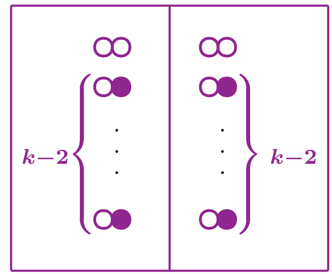
# Non-finite axiomatisability

For every $k < \omega$ there are two finite grids:

$\boxed{\mathfrak{F}_k}$




$\boxed{\mathfrak{G}_k}$




- $\boxed{\mathfrak{F}_k}$ is **not** a p-morphic image of a rectangle
- $\boxed{\mathfrak{G}_k}$ is a p-morphic image of a square
- **if $2^{m+1} \leq k$ then with $m$ variables we can't tell $\mathfrak{F}_k$ and $\mathfrak{G}_k$ apart**

$\rightsquigarrow$ **neither $\mathsf{Eq}(\mathsf{sRdf}_2)$ nor $\mathsf{Eq}(\mathsf{sRdf}_2^{sq})$ can be axiomatised using finitely many variables**

# Explicit axioms via representation game

*Hirsch–Hodkinson 1997a*

- step-by-step build **representations** for countable algebras in **RA**, $\mathbf{CA}_n$, $\mathbf{Df}_n$
- can be described as a game $\boxed{\mathcal{G}_\omega(\mathfrak{A})}$ between $\forall$ and $\exists$:

  **$\mathfrak{A}$ is representable** iff **$\exists$ has a winning strategy in $\mathcal{G}_\omega(\mathfrak{A})$**

- "$\exists$ has a winning strategy" $\iff$ (infinitely many) **universal formulas**
- discriminator varieties $\rightsquigarrow$ **equational axiomatisations**
- are all these axioms canonical? NO, when $n \geq 3$

**same technique can be used to obtain explicit (infinite) axiomatisations for $\mathbf{Eq}(\mathbf{sRdf}_2)$ and $\mathbf{Eq}(\mathbf{sRdf}_2^{sq})$**

**are these axioms canonical??**

# Canonical axioms via complete representation game?

*Hirsch–Hodkinson 1997b*

- step-by-step build **complete representations** for countable atom-structures (for $\mathbf{RA}$, $\mathbf{CA}_n$)
- same technique can be used for $\mathbf{sDf_2}$:

  can be described as a game $\boxed{\mathcal{G}_\omega(\mathfrak{F})}$ between $\forall$ and $\exists$, step-by-step building homomorphisms from larger and larger **rectangles** to $\mathfrak{F}$

  $\mathfrak{F}$ **is a p-morphic image of a rectangle** iff $\exists$ **has a winning strategy in** $\mathcal{G}_\omega(\mathfrak{F})$

**can we describe this with canonical equations/formulas??**

# Axioms for elementarily generated logics via hybrid logic

*Hodkinson 2006*

$\boxed{\mathcal{C}}$ **elementary** class of relational structures

$\downarrow$

$\boxed{\Pi(\mathcal{C})}$ **FO pseudo-equational** theory of $\mathcal{C}$

$\downarrow$ **algorithmic**

$\boxed{\Phi_{\mathcal{C}}}$ $= \{\iota_\theta : \theta \in \Pi(\mathcal{C})\}$ — set of **pure hybrid** formulas

$\downarrow$ **algorithmic**

$\boxed{\Sigma_{\Phi_{\mathcal{C}}}}$ $= \bigcup_{\iota \in \Phi_{\mathcal{C}}} \Sigma_\iota$ — set of '**modal** approximants'

$\boxed{\textbf{Logic\_of}(\mathcal{C}) = \textbf{modal logic axiomatised by } \Sigma_{\Phi_{\mathcal{C}}}}$

**not necessarily canonical axioms**

# So how do we get a canonical axiomatisation?

$\boxed{\mathcal{C}_{bad} = \{\mathfrak{F} : \ \mathfrak{F} \text{ is a countable grid that is **not** a p-morphic image of a **rectangle**}\}}$

For every $\boxed{\mathfrak{F} \in \mathcal{C}_{bad}}$ we define a **Sahlqvist** equation $\boxed{\varphi_{\mathfrak{F}}}$ such that

- $\boxed{\varphi_{\mathfrak{F}}}$ is valid in every rectangle
- $\boxed{\neg\varphi_{\mathfrak{F}}}$ is satisfiable in $\boxed{\mathfrak{F}}$

$$\rightsquigarrow \quad \boxed{\mathbf{sRdf_2:}} \quad \boxed{\mathbf{sDf_2}} \ + \ \boxed{\varphi_{\mathfrak{F}}} \text{ for all } \boxed{\mathfrak{F} \in \mathcal{C}_{bad}}$$

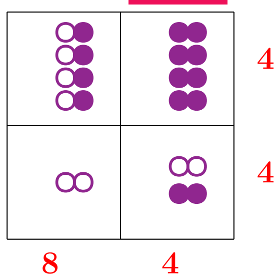
## ‘Finitary Sahlqvist reason’ $\varphi_{\mathfrak{F}}$ from a (possibly infinite) $\mathfrak{F}$?

There can be two kinds of reasons for a grid $\boxed{\mathfrak{F}}$ being bad:

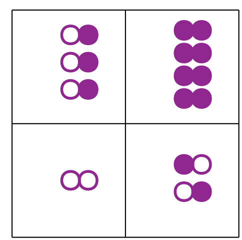
- either $\boxed{\mathfrak{F}}$ contains a <u>finite</u> bad bi-cluster
  (that itself is not a p-morphic image of a rectangle)
- or contains no such $\rightsquigarrow$ **linear constraint system** $\boxed{\Gamma^{\mathfrak{F}}}$ :
  - we consider the columns and rows in $\boxed{\mathfrak{F}}$ as variables
  - constraints come from the fact that the sizes of the rectangular p-morphic preimages of bi-clusters must ‘match’

but $\boxed{\Gamma^{\mathfrak{F}}}$ contains a <u>finite</u> ‘contradictory chain’ of constraints

**has solution** $\rightsquigarrow$ **good**




**no solution** $\rightsquigarrow$ **bad**



# Sahlqvist axiomatisation for $\mathbf{sRdf}_2^{sq}$ is much more complex

$\boxed{\mathbf{sRdf}_2^{sq}:}$ $\mathbf{sRdf}_2 +$

- **inductive equations** describing that

  if $\boxed{\xi}$ is a solution of $\boxed{\Gamma^{\mathfrak{F}}}$ then $\boxed{\sum_x \xi(x) = \sum_y \xi(y)}$

- the 'elsewhere' operations are **'reversive'**

  $\rightsquigarrow$ *Goranko–Vakarelov 2001*

  inductive equations are axiomatically equivalent to **Sahlqvist** equations

# Some mentioned papers


- D. Resek and R. Thompson, **Characterizing relativized cylindric algebras,** in: *Algebraic Logic* (eds.: H.Andréka, J.D.Monk, I.Németi), pp.245–292, North-Holland, 1991.
- R. Hirsch and I. Hodkinson, **Step by step – building representations in algebraic logic,** *Journal of Symbolic Logic,* 62:225–279, 1997.
- R. Hirsch and I. Hodkinson, **Complete representations in algebraic logic,** *Journal of Symbolic Logic,* 62:816–847, 1997.
- V. Goranko and D. Vakarelov, **Sahlqvist formulas in hybrid polyadic modal logics,** *Journal of Logic and Computation,* 11:737–754, 2001.
- I. Hodkinson and Y. Venema, **Canonical varieties with no canonical axiomatisation,** *Transactions of the American Mathematical Society,* 357:4579–4605, 2005.
- I. Hodkinson **Hybrid formulas and elementarily generated modal logics,** *Notre Dame Journal of Formal Logic,* 47:443–478, 2006.
- V. Goranko and D. Vakarelov, **Elementary canonical formulae: extending Sahlqvist's theorem,** *Annals of Pure and Applied Logic,* 141:180–217, 2006.
- R. Goldblatt and I. Hodkinson, **The McKinsey-Lemmon logic is barely canonical,** *The Australasian Journal of Logic,* 5:1–19, 2007.
- J. Bulian and I. Hodkinson, **Bare canonicity of representable cylindric and polyadic algebras,** *Annals of Pure and Applied Logic,* 164:884–906, 2013.
- S. Kikot, **A dichotomy for some elementarily generated modal logics,** *Studia Logica,* 103:1063–1093, 2015.