Temporal Logic of Minkowski Spacetime

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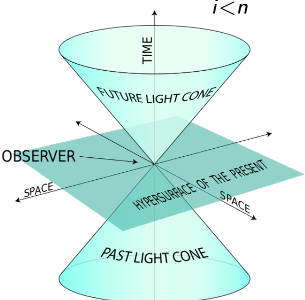
September 15, 2022

Minkowski Orders over $\mathbb{R}^{n+1}$

$n \geq 0$ spatial dimensions, one time dimension

$$(x_0, x_1, \ldots, x_{n-1}, x_n) < (y_0, y_1, \ldots, y_{n-1}, y_n) \iff$$
$$(x_n < y_n \wedge (y_n - x_n)^2 \geq \sum_{i<n} (y_i - x_i)^2)$$



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$$(x_0, x_1, \ldots, x_{n-1}, x_n) \leq (y_0, y_1, \ldots, y_{n-1}, y_n) \iff$$
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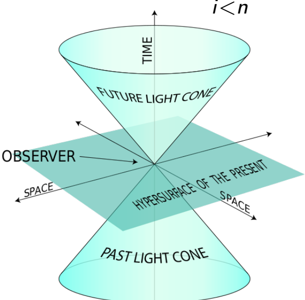

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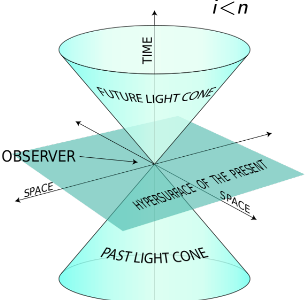
$n \geq 0$ spatial dimensions, one time dimension

$$(x_0, x_1, \ldots, x_{n-1}, x_n) \prec (y_0, y_1, \ldots, y_{n-1}, y_n) \iff (x_n < y_n \wedge (y_n - x_n)^2 > \sum_{i<n} (y_i - x_i)^2)$$



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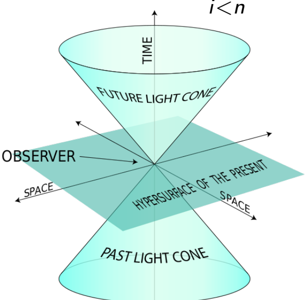
$$(x_0, x_1, \ldots, x_{n-1}, x_n) \rightarrow (y_0, y_1, \ldots, y_{n-1}, y_n) \iff$$
$$(x_n < y_n \wedge (y_n - x_n)^2 = \sum_{i<n} (y_i - x_i)^2)$$



Temporal Logic

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   - Decidability/Complexity

$F^{=}$, over $(\mathbb{R}^{n+1}, <)$

Let $k \geq 2$. Write $F^{=}(p_0, \ldots, p_{k-1})$ for

$$\bigwedge_{i<k} Fp_i \wedge G \bigwedge_{i \neq j < k} \neg(Fp_i \wedge Fp_j).$$

Asserts that each p_i is true in future at speed of light, but not less, in different directions.

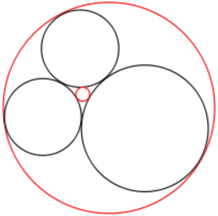
$(\mathbb{R}^{1+1}, <) \not\equiv (\mathbb{R}^{n+1}, <)$, for $n \geq 2$

$F^{=}(p_0, p_1, p_2)$ is satisfiable in $(\mathbb{R}^{n+1}, <)$ but not in $(\mathbb{R}^{1+1}, <)$.

Distinguishing higher dimensions?

Descartes Theorem:

In a plane at most four circles can meet pairwise tangentially.



Distinguishing higher dimensions?

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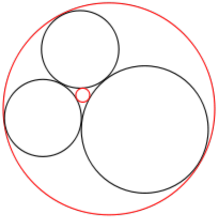
$$F^{=}(c_i : i < 4) \wedge$$

$$\Box \bigwedge_{i \neq j < 4} [(c_i \rightarrow F^{=}(t_{ij} : j \neq i < 4) \wedge (t_{ij} \rightarrow (P^{=}(c_i, c_j)]$$



Axioms for $(\mathbb{R}^{n+1}, <)$ $((\mathbb{R}^{n+1}, \leq))$?

1. Tautologies
2. $G(A \rightarrow B) \rightarrow (GA \rightarrow GB)$
3. $GA \rightarrow GGA$ (transitive)
4. $FGA \rightarrow GFA$ (confluent forward)
5. $A \rightarrow GPA$ (temporal)
6. $(F, G)/(P, H)$ duals

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5. $A \to GPA$ (temporal)
6. $(F, G)/(P, H)$ duals
7. [$GA \to A$ (reflexive)].
8. ?

Results for modal logic

$n \geq 1$ spatial dimensions.

$$\phi ::= prop | \neg\phi | (\phi \vee \phi') | F\phi | G\phi$$

order	axioms	complexity	citation
$\leq, \preceq$	S4.2 (Ax. 1–7)	PSPACE-complete	Goldblatt 1980
$<$	(Ax. 1–6)	PSPACE-complete	
$\prec$	OI.2	PSPACE-complete	Shehtman Shapirovsky 2002
$\rightarrow$	?	undecidable	Shapirovsky 2010

Temporal, $n = 0$ spatial dimensions

- Linear time, $(\mathbb{R}, \leq)$ or $(\mathbb{R}, <)$.
- finitely axiomatisable: transitive, linear, [reflexive]
- Decidable, NP-complete (by filtration).

Filtration

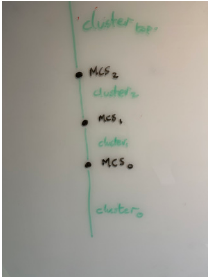
- $\mathsf{Cl}(\phi) = \{$subformulas, $\neg$subformulas$\}$.
- An MCS is a $\vdash$-consistent subset of $\mathsf{Cl}(\phi)$ containing exactly one of $\psi, \neg\psi$ for subformulas ψ.
- For $m, n \in$ MCS we let $m \lesssim n \iff$

$$\forall F\psi \in \mathsf{Cl}(\phi)\ (\psi / F\psi \in n \to F\psi \in m)$$
$$\wedge \forall P\psi \in \mathsf{Cl}(\phi)\ (\psi / P\psi \in m \to P\psi \in n).$$

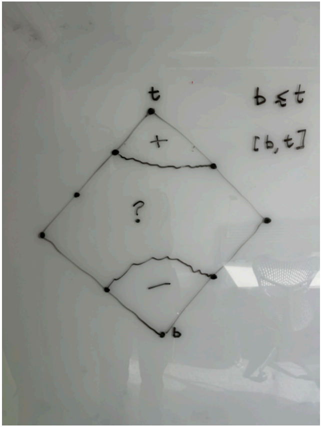
- Equivalence classes are called clusters.

Trace

$$(c_0 \lesssim m_0 \lesssim c_1 \lesssim \ldots, m_{k-1} \lesssim c_k), \text{ where } c_i \neq c_{i+1}$$



One spatial Dimension, boundary map

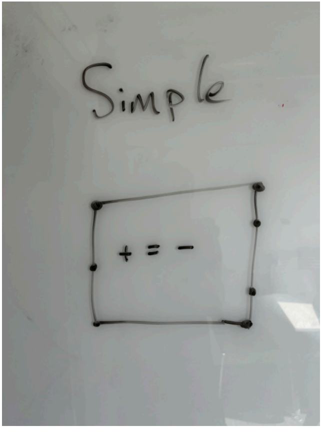


Fabricated boundary maps

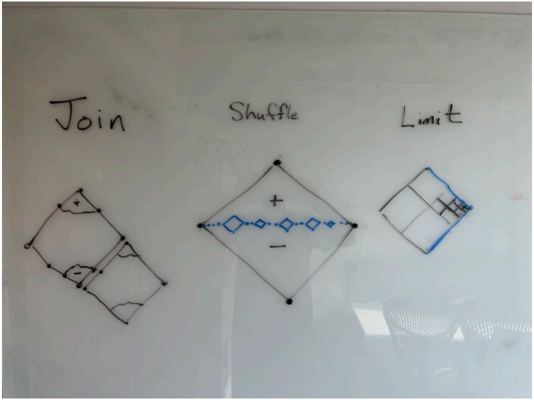
A boundary map is *fabricated* if

- Simple, i.e. $- = +$
- A join of two fabricated boundary maps
- A shuffle of fabricated bms, including a one-point bm
- A limit of fabricated bms.

Simple Boundary Map



Join, Shuffle, Limit



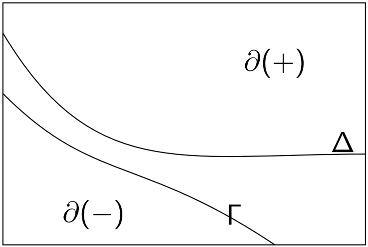
Fabricated equals Realisable

- Fabricated $\subseteq$ Realisable, fairly easy.
- Realisable $\subseteq$ Fabricated, more intricate.

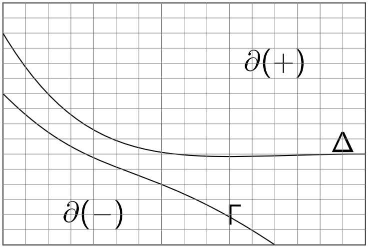
Fabricated $=$ Realisable

Realisable implies Fabricated, by induction on length of chain of distinct clusters from $\partial(-)$ to $\partial(+)$.
Base case: simple boundary maps.

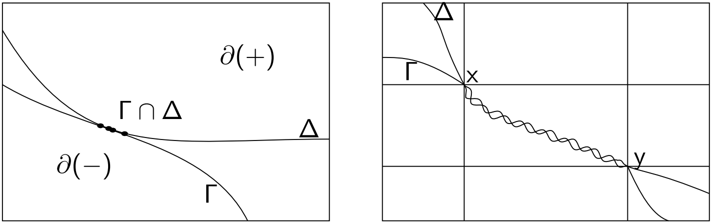
$\partial(-) < \partial(+)$ separated



$\partial(-) < \partial(+)$ separated



$\partial(-) < \partial(+)$ not separated



One spatial dimension

Satisfiability over $(\mathbb{R}^2, <)$, $(\mathbb{R}^2, \leq)$,
is PSPACE complete.

One spatial dimension

Satisfiability over $(\mathbb{R}^2, <)$, $(\mathbb{R}^2, \leq)$, $(\mathbb{R}^2, \prec)$, $(\mathbb{R}^2, \preceq)$ is PSPACE complete.

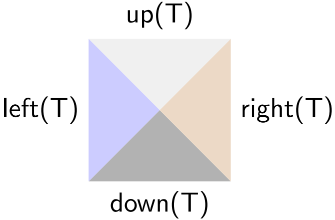
Corollary for Interval Logic

The temporal interval logic of the ‘during’ relation is PSPACE complete.

Satisfiability over $(\mathbb{R}^{n+1}, <)$ is EXTIME-hard, for $n \geq 2$

Satisfiability is EXPTIME-hard.
Find a suitable embedding of corridor tiling game tree into $(\mathbb{R}^{n+1}, <)$, for $n > 1$.

Wang tiles



2-player corridor-tiling game

Instance

$$((T_0, \ldots, T_{s+1}), (I_1, \ldots, I_n))$$

where the former are tile types, the latter a sequence of n tiles.
Alternating players: $\exists, \forall$, board $n \times \omega$ grid, row by row $\exists$ first in column 1
Winning conditions: Eloise if T_{s+1} played in 1st column; otherwise Abelard

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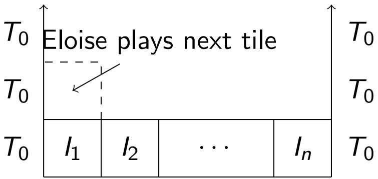
Decision problem: Does Eloise have a winning strategy?



Questions

- Axiomatise any of these temporal logics. Finitely?
- Distinguish frames by the number of spatial dimensions, or prove temporal equivalence.
- Precise complexity of satisfiability, with at least two spatial dimensions.

Selected bibliography

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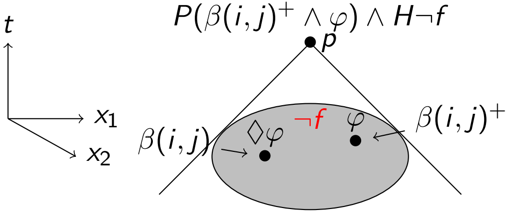
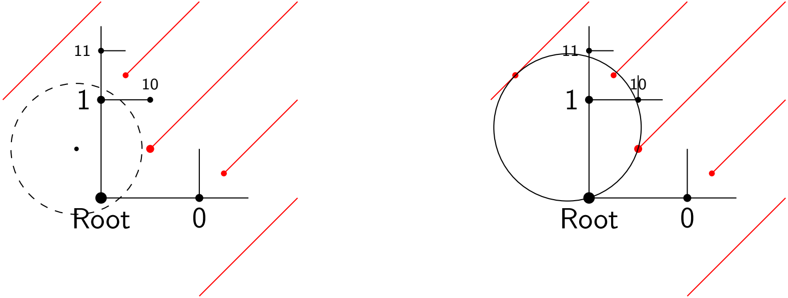
Auxilliary Modalities

$$\Diamond\varphi := \bigvee_{1\leq i\leq n,\ 0\leq j\leq b-1} \beta(i,j) \wedge F(P(\beta((i,j)^+) \wedge \varphi) \wedge H\neg f)$$



Drawing binary tree in Minkowski spacetime



Distinguishing Frames by Temporal Formulas

Frames		Distinguishing Fmla
valid	not valid	
$(\mathbb{R}^{n+1}, \leq)$	$(\mathbb{R}^{n+1}, <)$	$Gp \to p$
$(\mathbb{R}^2, <)$	$(\mathbb{R}^3, <)$	$\neg(\bigwedge_{i<3} Fp_i \wedge G \bigwedge_{i \neq j < 3} \neg(Fp_i \wedge Fp_j))$
$(\mathbb{R}^{n+1}, \prec)$	$(\mathbb{R}^{n+1}, <)$	$(Fp \wedge Fq) \to F(Fp \wedge Fq)$
$(\mathbb{R}^{n+1}, <)$	$(\mathbb{R}^{n+1}, \to)$	$Gp \to GGp$
$(\mathbb{R}^{n+1}, <)$	$(\mathbb{Q}^{n+1}, <)$	$(\neg Gp \wedge FGp) \to F(Gp \wedge H\neg Gp)$
$(\mathbb{R}^{n+1}, \leq)$	$(\mathbb{Q}^{n+1}, \leq)$	✓