Omitting types for finite variable fragments of first order logic

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Unless otherwise indicated n is fixed to be a finite ordinal > 2. Let $\mathfrak{L}$ be an extension or reduct or variant of first order logic, like first logic itself, or L_n with $2 < n < \omega$, $L_{\omega_1,\omega}$, etc.

An omitting types theorem for $\mathfrak{L}$, briefly an OTT, is typically of the form 'A countable family of non–isolated types in a countable $\mathfrak{L}$ theory T can be omitted in a countable model of T. From this it directly follows that if a type is realizable in every model of a countable theory T, then there should be a formula consistent with T that isolates this type.

A type is simply a set of formulas Γ say. The type Γ is realizable in a model if there is an assignment that satisfies (uniformly) all formulas in Γ.

Finally, ϕ isolates Γ means that $T \vdash \phi \to \psi$ for all $\psi \in \Gamma$. What Orey and Henkin proved is that the OTT holds for $L_{\omega,\omega}$ when such types are *finitary*, meaning that they all consist of n-variable formulas for some $n < \omega$.

OTT has an algebraic facet exhibited in the property of *atom-canonicity*; which in turn reflects an important *persistence* property in modal logic.

Algebraically, so–called *persistence properties* refer to closure of a variety V under passage from a given algebra $\mathfrak{A} \in \mathsf{V}$ to some 'larger' algebra $\mathfrak{A}^*$.

Canonicity, which is the most prominent persistence property in modal logic, the 'large algebra' $\mathfrak{A}^*$ is the canonical embedding algebra (or perfect) extension of $\mathfrak{A}$, a complex algebra based on the *ultrafilter frame* of $\mathfrak{A}$ whose underlying set is the set of all Boolean ultrafilters of $\mathfrak{A}$.

A completely additive variety of Boolea algebras with operators V is *atom-canonical*:

if whenever $\mathfrak{A} \in V$ is atomic, then the complex algebra of its atom structure, in symbols $\mathfrak{CmAt}\mathfrak{A}$, is also in V. More concisely, V is such if $\mathfrak{Cm}\mathsf{At}V \subseteq V$.

Atom-canonicity is concerned with closure under forming Dedekind-MacNeille completions (sometimes occuring in the literature under the name of *the minimal completions*) of atomic algebras in the variety V, because for an atomic $\mathfrak{A} \in V$, $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is its Dedekind-MacNeille completion.

Though RCA_n is canonical, it is not atom-canonical for $2 < n < \omega$ (to be proved in a while). From non-atom-canonicity of RCA_n, it follows that RCA_n cannot be axiomatized by Sahlqvist equations.

We shall see that (non-) atom-canonicity of subvarieties of RCA_n is closely related to (the failure) of some version of the OTT in modal fragments of L_n, such that the clique guarded fragment.

While the classical Orey-Henkin OTT holds for $L_{\omega,\omega}$, it is known that the OTT fails for L_n in the following (strong) sense. For every $2 < n \leq l < \omega$, there is a countable and complete L_n atomic theory T, and a single type, namely, the type consisting of co-atoms of T, that is realizable in every model of T, but cannot be isolated by a formula ϕ using l variables. Such ϕ will be referred to henceafter as a *witness*.

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Now we prove stronger negative OTTs for L_n when types are required to be omitted with respect to certain (much wider) generalized semantics, called m-*flat* and m–*square* with $2 < n < m < \omega$. Considering such *clique-guarded* semantics swiftly leads us to rich territory.

Roughly if we zoom in by a 'movable window' to an m-square representation,
there will come a point determined by m, where we mistake the m-square representation for an ordinary (ω-square) one. Our proofs are algebraic, demonstrating that, unlike CA_n itself, and like RCA_n, infinitely many varieties of CA_ns are not atom-canonical.

Blow up and blur constructions in connection to failure of OTTs:

This subtle construction may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive Boolean algebras with operators (BAOs). One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), blows it up, by splitting one or more of its atoms each to infinitely many subatoms, obtaining an (infinite) *countable* atomic $\mathfrak{B}b(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{B}b(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{B}b(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the Dedekind-MacNeille completion of $\mathfrak{B}b(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{B}b(\mathfrak{A})$.

Then any class $\mathbf{M}$ say, between $\mathbf{L}$ and $\mathbf{K}$ that is closed under forming subalgebras will not be atom–canonical, for $\mathfrak{B}b(\mathfrak{A}) \in \mathbf{L}(\subseteq \mathbf{M})$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{B}b(\mathfrak{A}) \notin \mathbf{K}(\supseteq \mathbf{M})$ because $\mathfrak{A} \notin \mathbf{M}$ and $\mathbf{SM} = \mathbf{M}$. We say, in this case, that $\mathbf{L}$ *is not atom–canonical with respect to* $\mathbf{K}$.

Let $2 < n \leq l < m \leq \omega$. We obtain negative results of the form $\Psi(l, m)$: *There exists a countable, complete and atomic* L_n *first order theory* T *in a signature* L, *meaning that the Tarski Lindenbuam quotient algebra* $\mathfrak{Fm}_T$ *is atomic, such that the type* Γ *consisting of co-atoms* $\mathfrak{Fm}_T$ *is realizable in every* m*–square model, but* Γ *cannot be isolated using* $\leq l$ *variables.*

An m*-square model of* T *is an* m*-square representation of* $\mathfrak{Fm}_T$. *We succeed to prove* $\Psi(n, m)$ *for all* $m \geq n(n+1)/2+1$ *and* $\Psi(l, \omega)$ *for all* $n < l < \omega$. *We say that* VT *fails almost everywhere.*

From now on, unless otherwise indicated, n is fixed to be a finite ordinal > 2.

Definition .1. An *n–dimensional atomic network* on an atomic algebra $\mathfrak{A} \in \mathsf{CA}_n$ is a map $N : {}^n\Delta \to \mathsf{At}\mathfrak{A}$, where Δ is a non–empty finite set of *nodes*, denoted by $\mathsf{nodes}(N)$, satisfying the following consistency conditions for all $i < j < n$:

(i) If $\bar{x} \in {}^n\mathsf{nodes}(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff \bar{x}_i = \bar{x}_j$,

(ii) If $\bar{x}, \bar{y} \in {}^n\mathsf{nodes}(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$,

Definition .2. Assume that $\mathfrak{A} \in \mathsf{CA}_n$ is atomic and that $m, k \leq \omega$. The *atomic game* $G^m_k(\mathsf{At}\mathfrak{A})$, *or simply* G^m_k, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds where $\forall$ is offered only one move, namely, *a cylindrifier move*: Suppose that we are at round $t > 0$. Then $\forall$ picks a previously played network N_t ($\mathsf{nodes}(N_t) \subseteq m$), $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $\bar{x} \in {}^n\mathsf{nodes}(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, $\exists$ has to deliver a network M such that $\mathsf{nodes}(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n\mathsf{nodes}(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$. We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G^m_k(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.

Lemma .3. *Let $2 < n < \omega$, and assume that $m > n$. If $\mathfrak{A} \in \mathsf{CA_n}$ is finite and $\mathfrak{A}$ has an m-square representation then $\exists$ has a winning strategy in $G^m_\omega(\mathsf{At}\mathfrak{A})$. In particular, if $\forall$ has a winning strategy in $G^m(\mathsf{At}\mathfrak{A})$, then $\mathfrak{A} \notin \mathsf{SNr}_n\mathsf{CA}_m$.*

Rainbow constructions:

Let G, R be two relational structures. Let $2 < n < \omega$. Then the colours used are:

- greens: g_i $(1 \leq i \leq n-2)$, g_0^i, $i \in \mathsf{G}$,
- whites : $\mathsf{w}_i : i \leq n-2$,
- reds: r_{ij} $: i < j \in n$,
- shades of yellow : $\mathsf{y}_S : S$ a finite subset of ω or $S = \omega$.

A *coloured graph* is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some $n-1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

Definition .4. Let $i \in \mathsf{G}$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M *an* i *- cone* if $M(x_0, z) = \mathrm{g}_0^i$ and for every $1 \leq j \leq n-2$, $M(x_j, z) = \mathrm{g}_j$, and no other edge of M is coloured green. $(x_0, \ldots, x_{n-2})$ is called the *base of the cone*, z the *apex of the cone* and i the *tint of the cone*.

The rainbow algebra depending on G and R from the class $\mathbf{K}$ consisting of all coloured graphs M such that:
M is a complete graph and M contains no triangles (called forbidden triples) of the following types:

$$(\mathsf{g}, \mathsf{g}^{'}, \mathsf{g}^{*}), (\mathsf{g}_i, \mathsf{g}_i, \mathsf{w}_i) \qquad , 1 \leq i \leq n-2, \quad (1)$$

$$(\mathsf{g}_0^j, \mathsf{g}_0^k, \mathsf{w}_0), j, k \in \mathsf{G}, \quad (2)$$

$$(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*}) \quad (3)$$

unless

$$|\{(j,k),(j',k'),(j^*,k^*)\}| = 3$$

and no other triple of atoms is forbidden.

Let G and R be relational structures as above. Take the set J consisting of all surjective maps $a : n \to \Delta$, where $\Delta \in \mathbf{K}$ and define an equivalence relation $\sim$ on this set relating two such maps iff they essentially define the same graph.

Let At be the atom structure with underlying set $J/\sim = \{[a] : a \in \mathrm{J}\}$. We denote the equivalence class of a by $[a]$. Then define, for $i < j < n$, the accessibility relations corresponding to ijth–diagonal element, and ith–cylindrifier, as follows:

(1) $[a] \in E_{ij}$ iff $a(i) = a(j)$,

(2) $[a]T_i[b]$ iff $a \upharpoonright n \setminus \{i\} = b \upharpoonright n \setminus \{i\}$,

These definitions are sound (do not depend on the representatives). Now consider the atom structure $\mathrm{At} = (J/\sim, T_i, E_{ij})_{i,j<n}$.

Let $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$ be the complex algebra over At. That is to say, the domain of $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$ is $\wp(\mathsf{At})$. The boolean operations are the usual set theoretic intersections and taking complements and the extra non boolean operations are defined for $X \subseteq \mathsf{At}$ as follows

$$\mathsf{c}_i X = \{[b] \in \mathsf{J} : \exists [a] \in X \ \ [a] T_i [b]\},$$

$$\mathsf{d}_{ij} = E_{ij}.$$

This, as easily checked, defines a CA_n atom structure. The complex CA_n over this atom structure will be denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$

For rainbow atom structures, there is a one to one correspondence between atomic networks and coloured graphs, so for $2 < n < m \leq \omega$, we use the graph versions of the games G_k^m, $k \leq \omega$, and $\mathbf{G}^m$ played on rainbow atom structures of dimension m.

The the atomic k rounded game game G_k^m where the number of nodes are limited to n to games on coloured graphs.

The typical winning strategy for $\forall$ in the graph version of both atomic games is bombarding $\exists$ with cones having a common base and *green* tints until she runs out of (suitable) *reds*, that is to say, reds whose indicies do not match.

Theorem .5. *1. Let $2 < n < \omega$ and $t(n) = n(n+1)/2+1$. The variety* RCA_n *is not-atom canonical with respect to* $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$.

2. In fact, there is a countable atomic simple $\mathfrak{A} \in \mathsf{RCA}_n$ *such that* $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ *does not have an* $t(n)$*-square,a fortiori* $t(n)$*- flat, representation.*

The proof is divided into four parts:

1: **Blowing up and blurring $\mathfrak{B}_f$ forming a weakly representable atom structure $\mathbf{At}$:** Take the finite rainbow CA$_n$, $\mathfrak{B}_f$ where the reds R is the complete irreflexive graph n, and the greens are $\{\mathrm{g}_i : 1 \leq i < n-1\} \cup \{\mathrm{g}_0^i : 1 \leq i \leq n(n-1)/2 + 2\}$, endowed with the cylindric operations. We will **blow up and blur** the atom structure of $\mathfrak{B}_f$, which we call $\mathbf{At}_f$; so that $\mathbf{At}_f = \mathsf{At}(\mathfrak{B}_f)$.

One then defines a larger the class of coloured graphs. Let $2 < n < \omega$. Then the colours used are like above except that each red is 'split' into ω many having 'copies' the form r^l_{ij} with $i < j < n$ and $l \in \omega$, with an additional shade of red ρ such that the consistency conditions for the new reds (in addition to the usual rainbow consistency conditions) are as follows:

1. $(r^i_{jk}, r^i_{j'k'}, r^{i^*}_{j^*k^*})$ unless $i = i' = i^*$ and

$$|\{(j,k),(j',k'),(j^*,k^*)\}| = 3$$

2. (r, ρ, ρ) and $(\mathsf{r}, \mathsf{r}^*, \rho)$, where r, r^* are any reds.

The consistency conditions can be coded in an $L_{\omega,\omega}$ theory T having signture the reds with ρ together with all other colours. The theory T is only a first order theory (not an $L_{\omega_1,\omega}$ theory) because the number of greens is finite..

One construct an n-homogeneous model M is as a countable limit of finite models of T using a game played between $\exists$ and $\forall$. In the rainbow game $\forall$ challenges $\exists$ with *cones* having green *tints* (g_0^i), and $\exists$ wins if she can respond to such moves. This is the only way that $\forall$ can force a win. $\exists$ has to respond by labelling *appexes* of two succesive cones, having the *same base* played by $\forall$.

By the rules of the game, she has to use a red label. She resorts to ρ whenever she is forced a red while using the rainbow reds will lead to an inconsistent triangle of reds. The number of greens make this strategy work. The winning strategy is implemented by $\exists$ using the red label ρ (a non-principal ultrafilter) that comes to her rescue whenever she runs out of 'rainbow reds'.,

2. **Representing a term algebra (and its completion) as (generalized) set algebras:** Having M at hand, one constructs two atomic n–dimensional set algebras based on M, sharing the same atom structure and having the same top element.

Deleting the one available red shade, set

$$W = \{\bar{a} \in {}^n\mathsf{M} : \mathsf{M} \models (\bigwedge_{i<j<n} \neg\rho(x_i, x_j))(\bar{a})\},$$

and for $\phi \in L^n_{\infty,\omega}$, let $\phi^W = \{s \in W : \mathsf{M} \models \phi[s]\}$. Here W is the set of all n–ary assignments in ${}^n\mathsf{M}$, that have no edge labelled by ρ.

Let $\mathfrak{A}$ be the relativized set algebra with domain $\{\varphi^W : \varphi$ a first-order L_n-formula$\}$ and unit W, endowed with the usual concrete operations read off the connectives.

Classical semantics for L_n *rainbow formulas* and their semantics by relativizing to W coincide *but not with respect to $L^n_{\infty,\omega}$ rainbow formulas*. Hence the set algebra $\mathfrak{A}$ is isomorphic to a cylinric set algebra of dimension n having top element ^{n}M, so $\mathfrak{A}$ is simple, in fact its Df reduct is simple.

Let $\mathfrak{E} = \{\phi^W : \phi \in L^n_{\infty,\omega}\}$ **with the operations defined like on** $\mathfrak{A}$ **the usual way**.

We have an isomorphism from $\mathfrak{Cm}\mathbf{At}$ to $\mathfrak{E}$ defined via $X \mapsto \bigcup X$. Since $\mathsf{At}\mathfrak{A} = \mathsf{At}\mathfrak{Tm}(\mathsf{At}\mathfrak{A})$, which we refer to only by $\mathbf{At}$, and $\mathfrak{Tm}\mathsf{At}\mathfrak{A} \subseteq \mathfrak{A}$, hence $\mathfrak{Tm}\mathsf{At}\mathfrak{A} = \mathfrak{Tm}\mathbf{At}$ is representable. The atoms of $\mathfrak{A}$, $\mathfrak{Tm}\mathsf{At}\mathfrak{A}$ and $\mathfrak{Cm}\mathsf{At}\mathfrak{A} = \mathfrak{Cm}\mathbf{At}$ are the coloured graphs whose edges are *not labelled* by ρ.

3. **Embedding** $\mathfrak{B}_f$ **into** $\mathfrak{Cm}\mathbf{At}$ **by mapping each atom the suprema of its subatoms which exist in the complete** $\mathfrak{Cm}\mathbf{At}$

Let CRG_f be the class of coloured graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. We can (and will) assume that $\mathsf{CRG}_f \subseteq \mathsf{CRG}$.

Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathsf{CGR}$. Here we identify a with $[a]$; no harm will ensue. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_b \sim N_a$, $(M, N \in \mathsf{CGR})$:

1. $a(i) = a(j) \iff b(i) = b(j)$,

2. $M_a(a(i), a(j)) = \mathsf{r}^l \iff N_b(b(i), b(j)) = \mathsf{r}^k$, for some $l, k \in \omega$,

and othersise identical.

Define the map Θ from $\mathfrak{B}_f = \mathfrak{Cm}\mathbf{At}_f$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathbf{At}_f$, via $M_a \mapsto \sum_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a.

4. $\forall$ **has a winning strategy in** $G^{t(n)}$At$(\mathfrak{B}_f)$**; and the required result:**

It is straightforward to show that $\forall$ has winning strategy first in the Ehrenfeucht–Fraïssé forth private game played between $\exists$ and $\forall$ on the complete irreflexive graphs $n+1(\leq n(n-1)/2+1)$ and n in $n+1$ rounds $\mathsf{EF}^{n+1}_{n+1}(n+1, n)$ since $n+1$ is 'longer' than n.

Using (any) $p > n$ many pairs of pebbles available on the board $\forall$ can win this game in $n+1$ many rounds.

$\forall$ lifts his winning strategy from the lst private Ehrenfeucht–Fraïssé forth game to the graph game on $\mathbf{At}_f = $ At$(\mathfrak{B}_f)$ forcing a win using $t(n)$ nodes.

One uses the $n(n-1)/2+2$ green relations in the usual way to force a red clique C, say with $n(n-1)/2+2$.

He needs $n-1$ nodes as the base of cones, plus $|P|+2$ more nodes, where $P=\{(i,j): i<j<n\}$ forming a red clique, triangle with two edges satisfying the same r_p^m for $p\in P$. Calculating, we get $t(n)=n-1+n(n-1)/2+2=n(n+1)/2+1$.

Then by the above Lemma, $\mathfrak{B}_f\notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$. . But $\mathfrak{B}_f$ embeds into $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, hence $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ is outside the variety $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{t(n)}$, as well.

Proof of the required

Using easy algebraic arguments we can prove $\Psi(n,m)$ **for all** $m \geq t(n) = n(n+1)/2+1$ **and** $\Psi(l,\omega)$ **for any** $2 < n < l < \omega$ **from the joint result with Andréka and Németi of constucting for all such** l **a countable atomic algebra** $\mathfrak{A} \in \mathsf{RCA}_c \cap \mathsf{Nr}_n\mathsf{CA}_l$ **such that** $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathsf{RCA}_n$**.**

Let $\mathbf{G}^m$ be the ω- rounded game using m nodes where $\forall$ has the option to reuse the m nodes in play.

Lemma .6. *Let* $2 < n < \omega$ *and* $n < m$. *If* $A \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ *then* $\exists$ *has a winning strategy in* $\mathbf{G}^m(\mathsf{At}\mathfrak{C})$

Theorem .7. *Any class* $\mathbf{K}$ *such that* $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathsf{K} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ *is not elementary*

1. Take the polyadic requality rainbow–like PEA_n, call it $\mathfrak{C}$, based on the ordered structure $\mathbb{Z}$ and $\mathbb{N}$. The reds R is the set $\{\mathsf{r}_{ij} : i < j < \omega(=\mathbb{N})\}$ and the green colours used constitute the set $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on $\mathbb{Z}$ and $\mathbb{N}$, but now the triple $(\mathsf{g}_0^i, \mathsf{g}_0^j, \mathsf{r}_{kl})$ is also forbidden if $\{(i,k),(j,l)\}$ is not an order preserving partial function from $\mathbb{Z} \to \mathbb{N}$.

2. It can be shown that $\forall$ has a winning strategy in the graph version of the game $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C})$ played on coloured graphs. The rough idea here, is that, as is the case with winning strategy's of $\forall$ in rainbow constructions, $\forall$ bombards $\exists$ with cones having distinct green tints demanding a red label from $\exists$ to appexes of succesive cones.

The number of nodes are limited but $\forall$ has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces $\exists$ to choose red labels, one of whose indices form a decreasing sequence in $\mathbb{N}$. In ω many rounds $\forall$ forces a win, so $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$.

3. I devise a game a k rounded game (stronger than G_k involving two amalgamation new moves) $\mathbf{H}_k$ such that if $\exists$ has a winning strategy in $H_k(\alpha)$, α an atom structure, then $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathsf{CA}_\omega$ $\alpha \in \mathsf{AtNr}_n\mathsf{CA}_\omega$. I show that $\exists$ has a winning strategy in $\mathbf{H}_k(\mathsf{At}\mathfrak{C})$ for all $k < \omega$. For each $k < \omega$, let σ_k describe the winning strategy of $\mathbf{H}_k(\alpha)$ where we assume $\mathfrak{C} = \mathfrak{Tm}\alpha$. Let $\mathfrak{D}$ be a non–principal ultrapower of $\mathfrak{C}$. Then $\exists$ has a winning strategy σ in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$

4. Now let $\mathfrak{B} = \bigcup_{i<\omega} \mathfrak{A}_i$. This is acountable elementary subalgebra of $\mathfrak{D}$, hence necessarily atomic, and $\exists$ has a winning strategy in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{B})$. So we get that $\mathfrak{CmAt}\mathfrak{B} \in \mathsf{Nr}_n\mathsf{CA}_\omega$. Since $\mathfrak{B} \subseteq_d \mathfrak{CmAt}\mathfrak{B}$, then $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$,$\mathfrak{B} \in \mathsf{CRCA}_n$. But $\forall$ has a winning strategy in $\mathbf{G}^m(\mathsf{At}\mathfrak{B})$, $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$. To finalize the proof, let $\mathbf{K}$ be as given. Then $\mathfrak{B} \equiv \mathfrak{C}$, $\mathfrak{B} \in \mathbf{K}(\subseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n)$, but $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}(\supseteq \mathbf{K})$ giving that $\mathbf{K}$ is not elementary.