Lorentzian Structures on Branching Spacetimes

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Main Talking Point

By suitably modifying the standard Belnappian theory of Branching Spacetimes, one can naturally endow the resulting models with appropriate topological, differentiable and Lorentzian structures.

This procedure can be generalised so as to define a new class of branching spacetimes known as Lorentzian Branching Spacetimes.

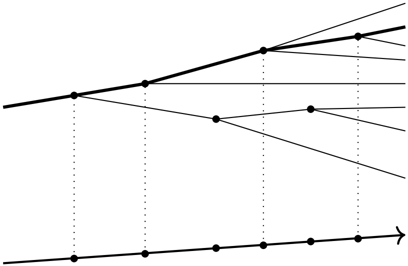
Branching Spacetimes = Branching + Spacetimes

Overview

Preliminaries:

Branching Spacetimes $\neq$ Branching + Spacetimes

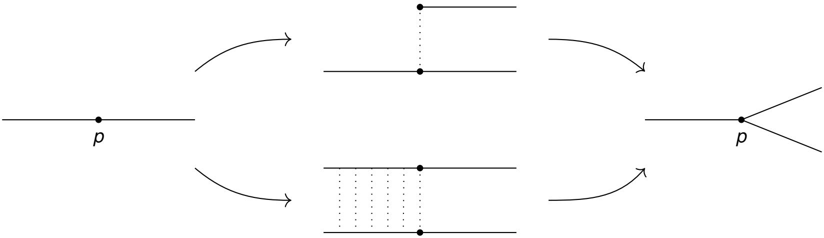
Branching Temporal Models



Branching is enabled by globally relaxing a property of the temporal ordering $\leq$. *Histories* are then defined to be maximal subsets retaining this globally-relaxed property.

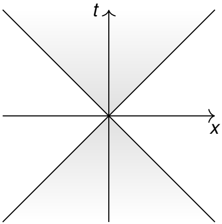
Constructing a Branching Model

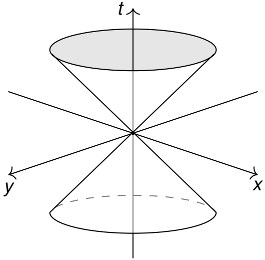
There are two main ways to turn a linear temporal model into a branching one.



1. Glue on another future at p.
2. Take two copies of the model, and glue everywhere outside of the future of p.

Minkowski Spacetime





A vector space $\mathbb{R}^n$, together with a *pseudometric* η. The metric η acts on elements of $\mathbb{R}^n$ by:

$$\eta(x, y) = -x_0 y_0 + x_1 y_1 + ... + x_{n-1} y_{n-1}$$

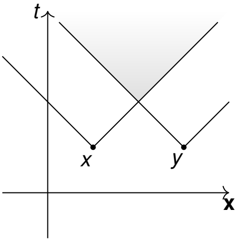
The minus sign is needed to ensure that the speed of light is constant for all observers.

Causal Structure of Minkowski Spacetime

We can use lightcones to define a binary relation $\leq^\eta$ that encodes the causal structure of M^n, by saying that $x \leq^\eta y$ iff y lies in the future lightcone of x.

Lemma (Causal Properties of M^n)

The ordering $\leq^\eta$ is a dense, directed partial order in which each upper-bounded chain has a supremum, and each lower-bounded chain has an infimum.

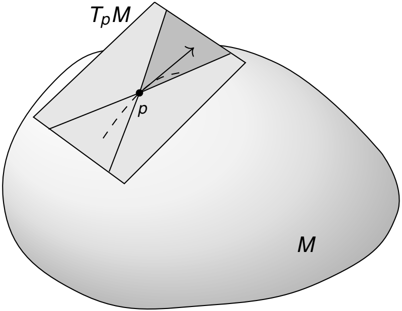


What is a Spacetime?

Definition

A spacetime is a connected smooth manifold M together with a Lorentzian metric g and a time-orientation.

Geometric	$(M, \tau, \mathcal{A}, g)$
Differentiable	$(M, \tau, \mathcal{A})$
Topological	(M, τ)
Set Theoretic	M
(Logical)	



The Theory BST92

The primitives of BST92 are a set W and a binary relation $\leq$.

BST1 The tuple $(W, \leq)$ is a dense partial order with no maxima.

Observe that W is not required to be directed. Histories of W are then defined to be *maximally directed subsets of* W.

BST2 If C is a lower-bounded chain of W then C has an infimum in W, which we denote by $inf(C)$.

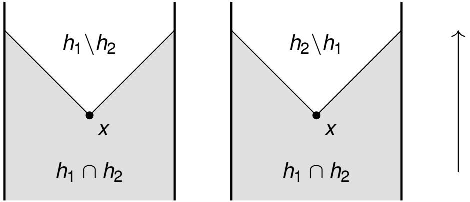
BST3 If C is an upper-bounded chain of W then C has a suprema in every history h such that $C \subseteq h$. We denote such a supremum by $sup_h(C)$.

The Prior Choice Principle

The last axiom deals with the branching structure of BSTs.

PCP (Prior Choice Principle) If C is a chain of W such that $C \subseteq h_1 \backslash h_2$, then there is some element x of W such that $x \leq C$ and x is maximal in $h_1 \cap h_2$.

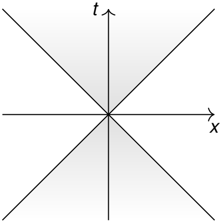
Such an x is called a *choice point* for h_1 and h_2, and is the last point contained within the overlap $h_1 \cap h_2$.



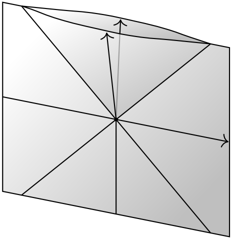
Minkowskian Branching Spacetimes

Definition

A BST92 model $(W, \leq)$ is called a *Minkowskian BST* iff each history of W is order-isomorphic to some fixed $(M^n, \leq^\eta)$.



The Minkowski spacetime M^2



The simple MBST M_2^2

Branching Spacetimes $\neq$ Branching + Spacetimes

There are two obvious limitations of BST92 models:

(L1) Models of BST92 only axiomatise the causal structure of spacetimes, so are too coarse to be interpreted as models of relativity.

(L2) BST92 can only deal with special-relativistic branching, and does not treat the models of *general* relativity adequately.

The State of the Art

Structure	Minkowski Spacetime	
	Standard	Branching under BST92
Underlying Set	M^n, i.e. $\mathbb{R}^n$	MBSTs
Causal Order	$\leq^\eta$	Ordering on MBSTs
Topology	Standard topology	Bartha topology*
Smooth Structure	Standard structure	??
Metric Structure	η	??

*The Bartha topology τ_B^W has been proposed as a natural topological extension of the order-theoretic models $(W, \leq)$.

Our Approach:

Modify BST92 and construct some new models

Our Approach

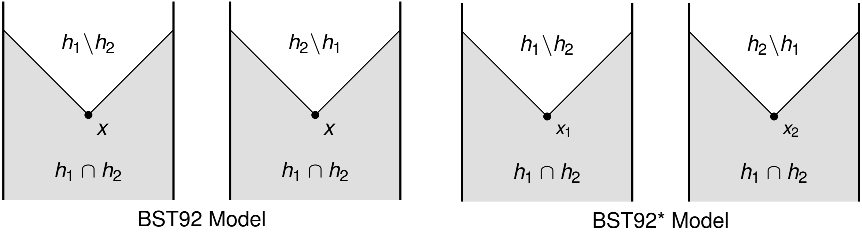
We remove the limitations L1 and L2 by:

1. Developing the mathematical machinery required to construct Minkowskian BSTs at the level of (M^n, η) (as opposed to the standard construction, which is at the level of the causal structure $(M^n, \leq^\eta)$.
2. Using this same machinery to form BSTs from arbitrary spacetimes (M, g).

This is done using a modified theory BST92*, and opting for a *type-2* construction of its MBSTs.

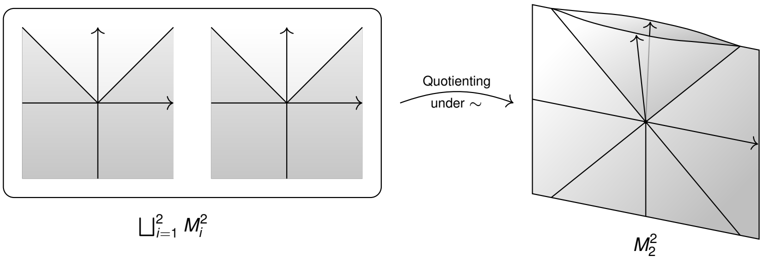
Models of BST92*

BST92* is mostly the same as BST92, except that it uses *choice pairs* instead of choice points. The elements of a choice pair are now distinct, $\leq$-*minimal* elements of the differences $h_1 \backslash h_2$ and $h_2 \backslash h_1$.



An Example: Constructing M_2^2

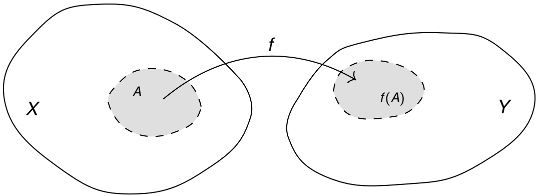
We can construct our simple MBST using this technique.



Here $(x, i) \sim (y, j)$ iff $x = y$ and $x \notin J^+(0)$.

Adjunction Spaces in Topology

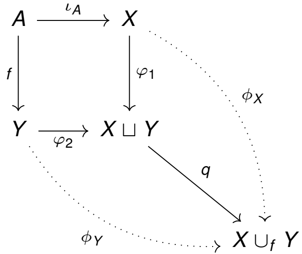
The idea of quotienting a disjoint union is a well-known construction in topology, known as an *adjunction space*.

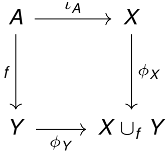


The information on where to glue the topological spaces X and Y together is formalised using a closed subspace A of X and a continuous map $f : A \to Y$. The adjunction space $X \cup_f Y$ is formed by quotienting the disjoint union $X \sqcup Y$ under the relation that identifies every $(a, 1)$ with $(f(a), 2)$.

Diagram for an Adjunction Space

The configuration for an adjunction space can be represented as:





Here the φ_i are the canonical injections into the disjoint union, and q is the quotient map associated to the identification of A and $f(A)$. The maps ϕ_X and ϕ_Y are called *canonical maps*, and are defined as the compositions $\phi_X := \varphi_1 \circ q$ and $\phi_Y := \varphi_2 \circ q$.

Adjoined Spacetimes

Theorem (Main Result)

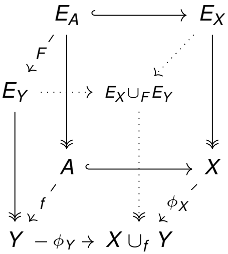
Let X and Y be spacetimes of the same dimension, such that:

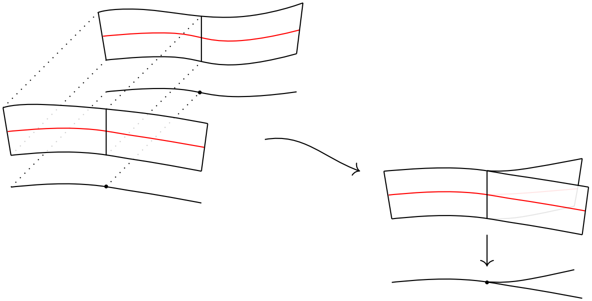
1. *each A is an open sub-spacetime of X, and*
2. *the map $f : A \to Y$ is a time-orientation-preserving isometric embedding.*

Then the adjunction space $X \cup_f Y$ can be equipped with a Lorentzian metric $\tilde{g}$ and a time-orientation $\tilde{\mathcal{T}}$, turning it into a spacetime. Moreover, the canonical maps ϕ_X and ϕ_Y act as open isometric embeddings that preserve time-orientation.

Key Idea: If we assert that f preserves all of the structure of A, then the gluing procedure does not destroy any information.

Sketch of the Proof





Key Results:

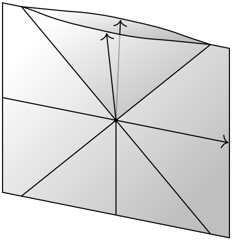
Branching Spacetimes = Branching + Spacetimes

Expressing MBSTs as Adjoined Spacetimes

We can construct the MBST M_2^2 by taking:

- X and Y equal to (M^n, η),
- the subspace A to be equal to $M^n \backslash J^+(0)$, and
- each $f : A \to Y$ is the inclusion map.

This data meets the criteria of the previous theorem, so we can conclude that the adjunction system $X \cup_f Y$ is a smooth manifold possessing a Lorentzian metric $\tilde{\eta}$ and a time orientation $\tilde{\mathcal{T}}$, that is, $X \cup_f Y$ is a (non-Hausdorff) spacetime!

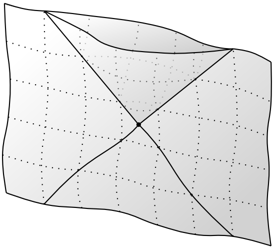


Defining Lorentzian BSTs

Now let (M, g) be any spacetime. We take:

- X and Y equal to (M, g),
- the subspace A to be equal to $M \setminus Cl(J^+(p))$, and
- each $f : A \to Y$ is the inclusion map.

Again this data meets the criteria of the previous theorem, so we get the following picture emerge:



Causal Properties of LBSTs

Theorem

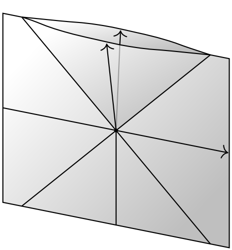
Let (M, g) be a Hausdorff spacetime, and $M_{\mathcal{C}}$ a Lorentzian BST built from M.

1. *If M is causal, then so is $M_{\mathcal{C}}$.*
2. *If M has a global time function, then so does $M_{\mathcal{C}}$.*
3. *If M has compact causal diamonds, then so does $M_{\mathcal{C}}$.*

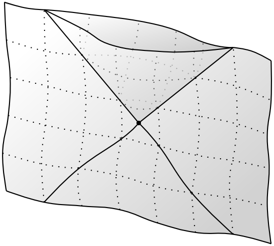
Back to the Limitations

(L1) Models of BST92 only axiomatise the causal structure of spacetimes, so are too coarse to be interpreted as models of relativity.

(L2) BST92 can only deal with special-relativistic branching, and does not treat the models of *general* relativity adequately.



Minkowskian BST



Lorentzian BST

The End

Useful Resources

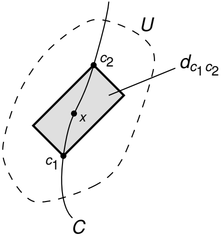
1. R. Penrose, “Techniques of differential topology in relativity”. Siam, 1972, vol. 7.
2. N. Belnap “Branching space-time”, *Synthese*, vol. 92, no. 3, pp. 385434, 1992.
3. T. Müller, “Branching space-time, modal logic and the counterfactual conditional”, in *Non-locality and modality*, Springer, 2002, pp. 273291.
4. T. Müller, “A generalized manifold topology for branching space-times”, *Philosophy of science*, vol. 80, no. 5, pp. 10891100, 2013.
5. T. Placek, N. Belnap, and K. Kishida, “On topological issues of indeterminism”, *Erkenntnis*, vol. 79, no. 3, pp. 403436, 2014.

The Bartha Topology on BST92 Models

The Bartha topology τ_B^W on a BST92 model $(W, \leq)$ is defined as follows.

Definition

A subset U of W is open in the Bartha topology iff for all x in U and all maximal $\leq$-chains C passing through x, there are elements c_1 and c_2 such that $x \in d_{c_1 c_2} \subset U$.



Comparison of Bartha Topologies

As it turns out, the Bartha topology on BST92* models is much better behaved.

Properties	BST92	BST92*
Histories open	No†	Yes
$\tau_B^h \subseteq \tau_B^W$	No†	Yes
$\tau_B^h = \tau_S^h$	Unknown	Yes
Hausdorff	No†	No†
Locally-Euclidean	No*	Yes (in MBSTs)
Connected	Yes	Yes
Path-connected	Unknown	Yes
Natural Extension	Yes	Yes

($\dagger$) In general no, but yes iff W is a single-historied model.
($*$) Yes iff W is equal to $(M^n, \leq^\eta)$.