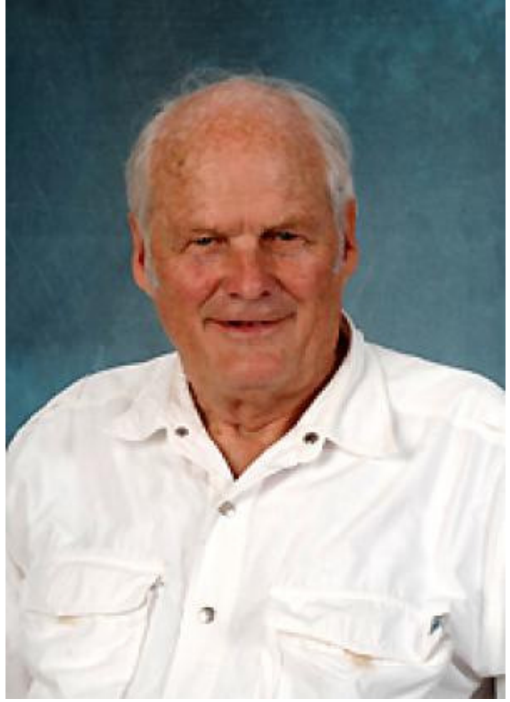
CONCEPT ALGEBRA OF RELATIVISTIC SPACETIME

Judit Madarász

joint research with

Hajnal Andréka, István Németi

Gergely Székely



An introduction to cylindric set algebras

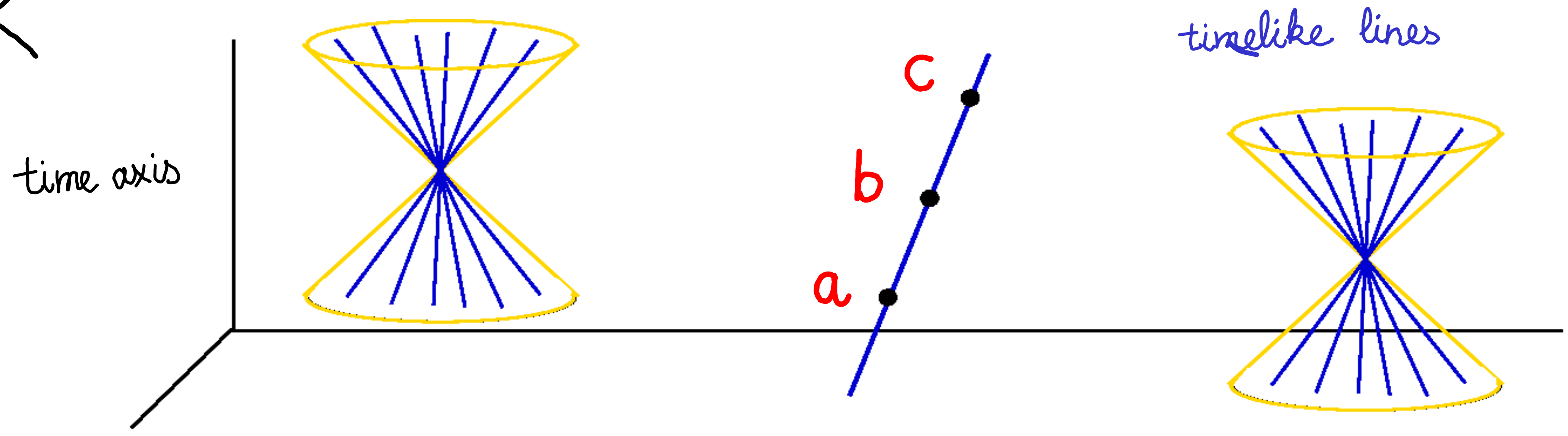
JD Monk

Logic Journal of the IGPL, Volume 8, Issue 4, July 2000, Pages 451–496, https://doi.org/10.1093/jigpal/8.4.451
Published: 01 July 2000

The corresponding facit of the theory of cylindric algebras is to describe the cylindric set algebras $\mathfrak{Cs}\,\mathfrak{M}$ for important models $\mathfrak{M}$. This amounts to looking at complete theories only, which is customary in model theory. It is somewhat surprising that this aspect of the theory of cylindric algebras has been almost entirely neglected. A complete description of $\mathfrak{Cs}\,\mathfrak{M}$ is known only in the case in which $\mathfrak{M}$ has only one-place relations. There are many other simple structures where the description of $\mathfrak{Cs}\,\mathfrak{M}$ should not be difficult; for example, for $\mathfrak{M}$ the rationals under their natural ordering.

SPECIAL RELATIVISTIC SPACETIME

$\mathbb{R}^4$



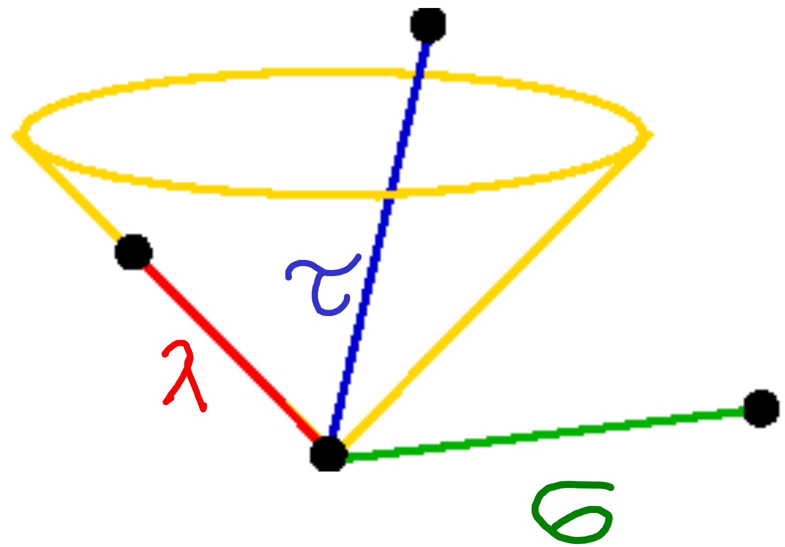
timelike collinearity $col^T(a,b,c) \iff a,b,c$ are on a timelike line

Relativistic Spacetime $RS = \langle \mathbb{R}^4, col^T \rangle$

We work in FOL

We will see that from timelike collinearity col^T one can define the full-fledged scale invariant Minkowski spacetime

$$\left.\begin{array}{lll} \text{timelike} & \text{connectedness} & \tau \\ \text{lightlike} & \text{——}\|\text{——} & \lambda \\ \text{spacelike} & \text{——}\|\text{——} & \sigma \end{array}\right\} \text{binary relations}$$



Minkowski equidistance, orthogonality etc

Definable by an open formula in FOL

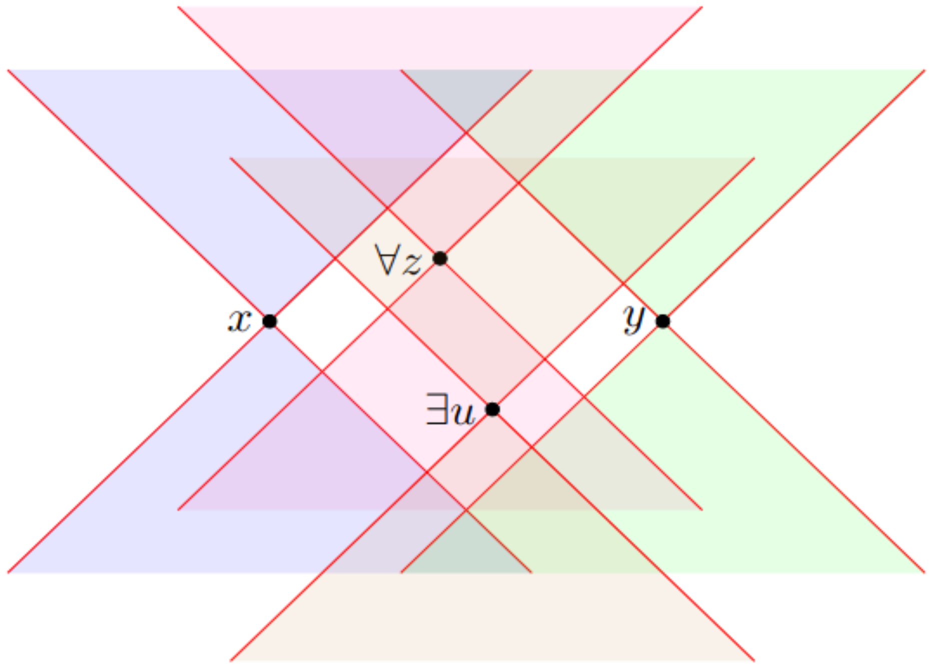
$$a \,\tau\, b \iff a \neq b \wedge \exists c \; col^T(a,b,c)$$

Definition of spacelike relatedness σ from timelike relatedness τ using 4 variables. (works for arbitrary ordered fields)

$\bar{\tau}$ $\neg\tau$

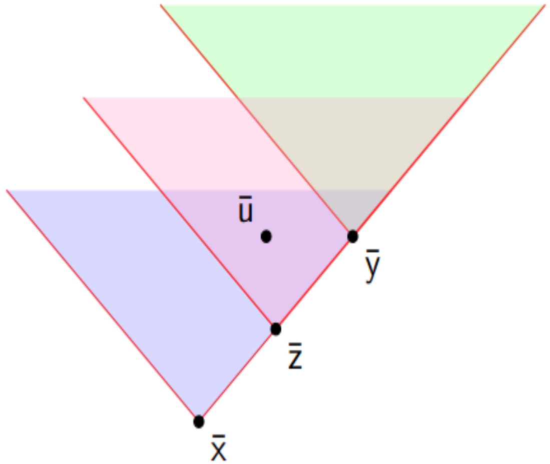
$$\Psi_{\tau\to\sigma}(x,y) \stackrel{\text{def}}{=} x \neq y\,,\, \forall z\big(z = x \vee z = y \vee \exists u(u\,\tau\, z\,,\, u\,\bar{\tau}\, x\,,\, u\,\bar{\tau}\, y)\big).$$

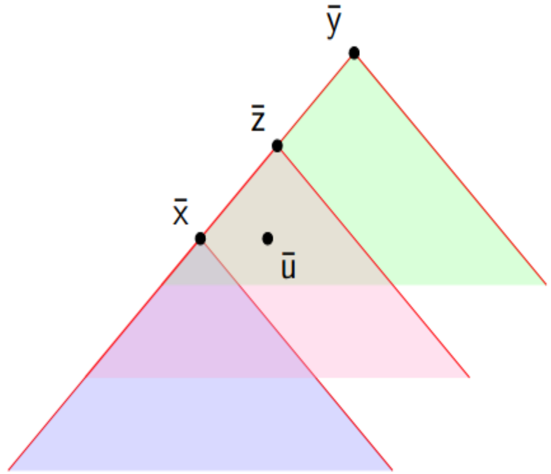
x and y are distinct points and inside the lightcone of every point z (different from x and y) there is a point u which is neither inside the lightcone of x nor that of y



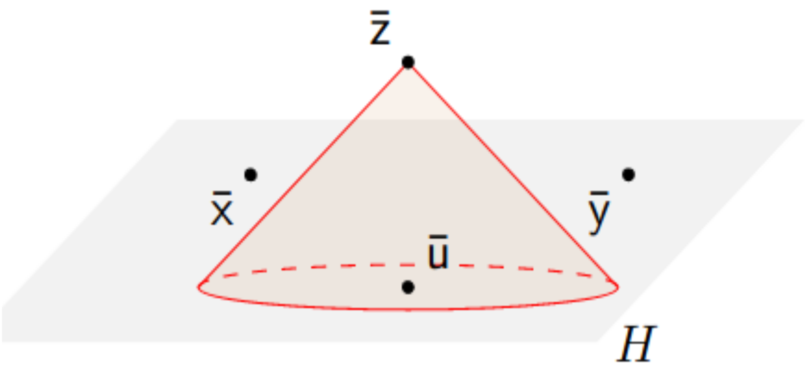
$$\Psi_{\tau\to\sigma}(x,y) \overset{\text{def}}{=} x \neq y\,,\ \forall z\big(z = x \vee z = y \vee \exists u(u\;\tau\;z\,,\ u\;\bar{\tau}\;x\,,\ u\;\bar{\tau}\;y)\big).$$

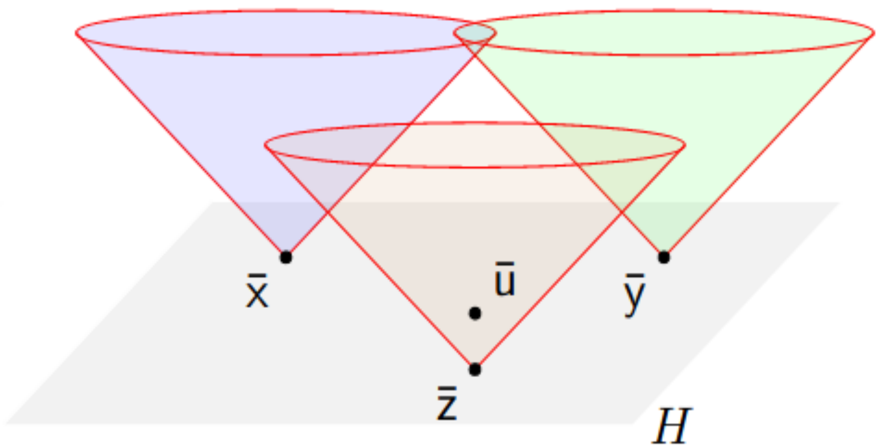
Proof: $\Psi_{\tau\to\sigma}(x,y) \Longrightarrow x\,\sigma\,y$





$x\,\sigma\,y \Longrightarrow \Psi_{\sigma\to\tau}(x,y)$





COMPLEXITY IN THE INTERDEFINABILITY OF TIMELIKE, LIGHLIKE AND SPACELIKE RELATEDNESS RELATIONS OF MINKOWSKI SPACETIME

HAJNAL ANDRÉKA, JUDIT X. MADARÁSZ, ISTVÁN NÉMETI, GERGELY SZÉKELY

$$\Psi_{\sigma\to\tau}(x,y) \overset{\text{def}}{=} x \neq y \,,\, \forall z\big(z = x \vee z = y \vee \exists u(u\ \sigma\ z \,,\, u\ \bar{\sigma}\ x \,,\, u\ \bar{\sigma}\ y)\big).$$

$$\Psi_{\lambda\to\sigma}(x,y) \overset{\text{def}}{=} \exists z\big(x\ \lambda\ z \,,\, \neg\exists u(u\ \lambda\ x \,,\, u\ \lambda\ y \,,\, u\ \lambda\ z)\big).$$

$$\Psi_{\sigma\to\lambda}(x,y) \overset{\text{def}}{=} \neg\Psi_{\sigma\to\tau}(x,y) \,,\, x\ \bar{\sigma}\ y \,,\, x \neq y.^{6}$$

$$\mathcal{W}_{\sigma\to\lambda}(x,y) \overset{\text{def}}{=} x\ \bar{\sigma}\ y \,,\, x \neq y \,,\, \forall u \forall v \exists z_u \exists z_v (z_u\ \bar{\sigma}\sigma\sigma\ uxy \vee z_v\ \bar{\sigma}\sigma\sigma\ vxy \vee u\ \bar{\sigma}\ v)$$

$$\mathcal{W}_{\sigma\to\tau}(x,y) \overset{\text{def}}{=} \neg\mathcal{W}_{\sigma\to\lambda}(x,y) \,,\, x\ \bar{\sigma}\ y \,,\, x \neq y.$$

$$\Psi_{\lambda\to\tau}(x,y) \overset{\text{def}}{=} \neg\Psi_{\lambda\to\sigma}(x,y) \,,\, x\ \bar{\lambda}\ y \,,\, x \neq y.$$

$$\mathcal{E}_{\sigma\to\tau}(p,q) \overset{\text{def}}{=} \exists r \exists x \exists s \exists z(r\ \sigma\sigma\ pq \,,\, x\ \sigma\bar{\sigma}\ pq \,,\, s\ \bar{\sigma}\bar{\sigma}\ pq \,,\, z\ \bar{\sigma}\sigma\ pq \,,\, r\ \bar{\sigma}\sigma\bar{\sigma}\ xsz)$$

$$\Psi_{\tau\to\lambda}(x,y) \overset{\text{def}}{=} \neg\Psi_{\tau\to\sigma}(x,y) \,,\, x\ \bar{\tau}\ y \,,\, x \neq y.$$

$$\hat{\mathcal{U}}_{\sigma\to\lambda} \overset{\text{def}}{=} \neg\mathcal{E}_{\sigma\to\tau}(p,q) \,,\, p \neq q \,,\, p\ \bar{\sigma}\ q.$$

$$\hat{\mathcal{E}}_{\sigma\to\tau} \overset{\text{def}}{=} \exists x \exists y \exists z(x\ \sigma\sigma\bar{\sigma}_{\neq}\bar{\sigma}_{\neq}\ pyqz \,,\, y\ \bar{\sigma}_{\neq}\bar{\sigma}_{\neq}\sigma\ pqz \,,\, z\ \bar{\sigma}_{\neq}\sigma\ pq)$$

$$\hat{\mathcal{E}}_{\tau\to\sigma} \overset{\text{def}}{=} \exists x \exists y \exists z(x\ \tau\tau\bar{\tau}_{\neq}\bar{\tau}_{\neq}\ pyqz \,,\, y\ \bar{\tau}_{\neq}\bar{\tau}_{\neq}\tau\ pqz \,,\, z\ \bar{\tau}_{\neq}\tau\ pq).$$

$$\mathcal{E}_{\tau\to\sigma}(p,q) \overset{\text{def}}{=} \exists r \exists x \exists s \exists z(r\ \tau\tau\ pq \,,\, x\ \tau\bar{\tau}\ pq \,,\, s\ \bar{\tau}\bar{\tau}\ pq \,,\, z\ \bar{\tau}\tau\ pq \,,\, r\ \bar{\tau}\tau\bar{\tau}\ xsz),$$

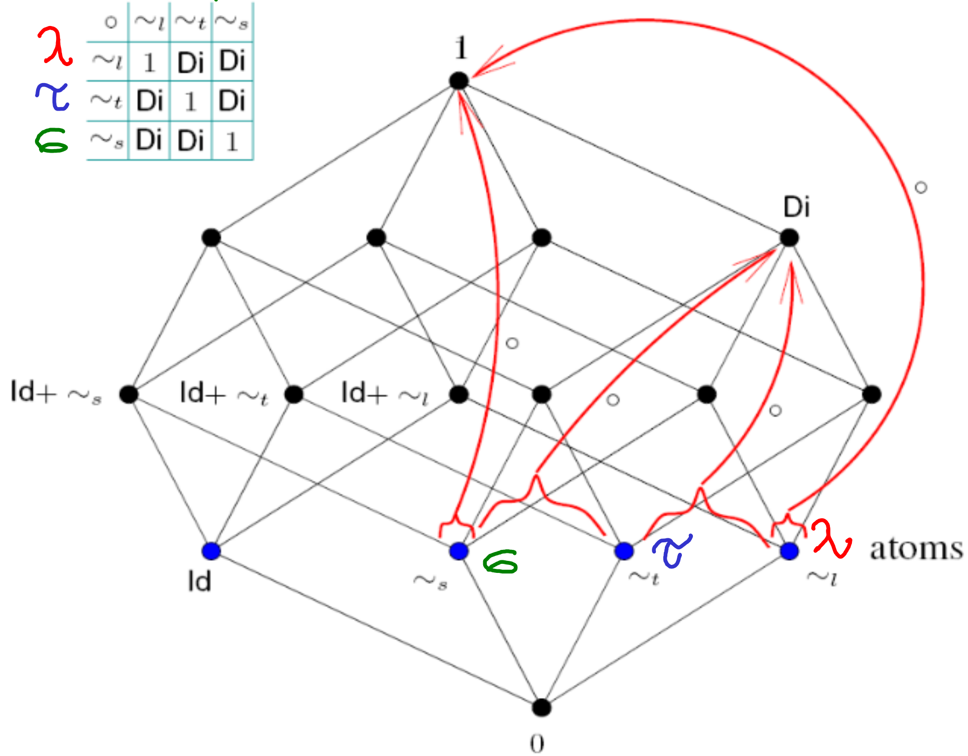
$n>2$, (Eucl.)	$\tau\to\sigma$	$\tau\to\lambda$	$\sigma\to\tau$	$\sigma\to\lambda$	$\lambda\to\tau$	$\lambda\to\sigma$
$\exists^2$ or $\forall^2$	$\nexists$ (not possible)					
$\exists^3$	$\hat{\mathcal{E}}_{\tau\to\sigma}$	$\nexists$	?	$\nexists$ (not possible)		
$\forall^3$	$\nexists$	$\hat{\mathcal{U}}_{\tau\to\lambda}$	$\nexists$	?	$\nexists$	
$\exists^4$	$\mathcal{E}_{\tau\to\sigma}$	$\nexists$	?	$\nexists$ (not possible)		
$\forall^4$	$\nexists$	$\mathcal{U}_{\tau\to\lambda}$	$\nexists$	?	$\nexists$	
$\exists^*$	✓	$\nexists$	?	$\nexists$ (not possible)		
$\forall^*$	$\nexists$	✓	$\nexists$	?	$\nexists$	
$\exists^1\forall^1$	?	$\Psi_{\tau\to\lambda}$	?	$\Psi_{\sigma\to\lambda}$	?	$\Psi_{\lambda\to\sigma}$
$\forall^1\exists^1$	$\Psi_{\tau\to\sigma}$	?	$\Psi_{\sigma\to\tau}$	?	$\Psi_{\lambda\to\tau}$	?
$\exists^2\forall^1$ / $\exists^1\forall^2$	?	✓	?	✓	?	✓
$\forall^2\exists^1$ / $\forall^1\exists^2$	✓	?	✓	?	✓	?
$\exists^2\forall^2$	?	✓	$\mathcal{W}_{\sigma\to\tau}$	✓	?	✓
$\forall^2\exists^2$	✓	?	✓	$\mathcal{W}_{\sigma\to\lambda}$	✓	?
$\exists^*\forall^*$	?	✓	✓	✓	?	✓
$\forall^*\exists^*$	✓	?	✓	✓	✓	?

$$\Psi_{\tau\to\sigma}(x,y) \stackrel{\mathsf{def}}{=} x \neq y\,,\ \forall z\big(z = x \vee z = y \vee \exists u(u\ \tau\ z\,,\ u\ \bar{\tau}\ x\,,\ u\ \bar{\tau}\ y)\big).$$

Thm Over any ordered field, from any of the relations τ, σ, λ, it is not possible to define any other from the same triple using only 3 variables.

Proof idea: Algebraic logic. In the algebra of binary relations none of the relations τ, λ and σ generates any other one of them.

		λ	τ	σ
	$\circ$	$\sim_l$	$\sim_t$	$\sim_s$
λ	$\sim_l$	1	Di	Di
τ	$\sim_t$	Di	1	Di
σ	$\sim_s$	Di	Di	1



Definitionally equivalent structures

$$\langle \mathbb{R}^4, col^T \rangle \equiv_\Delta \langle \mathbb{R}^4, \tau \rangle \equiv_\Delta \langle \mathbb{R}^4, \sigma \rangle \equiv_\Delta \langle \mathbb{R}^4, \lambda \rangle$$

in classical sense

Same relations can be defined

contain the same concepts

CONCEPTS

CONCEPT ALGEBRAS

Def (Concepts)

A **concept** in FOL model $\mathcal{M}$ is the extension of any open formula (a defined relation) If $\varphi(x_1, \dots, x_n)$ is a formula in the language of $\mathcal{M}$ with free variables $x_1, \dots, x_n$, its extension in $\mathcal{M}$ is

$$\varphi(x_1, \dots, x_n)^{\mathcal{M}} = \{\langle a_1, \dots, a_n \rangle : \mathcal{M} \models \varphi(a_1, \dots, a_n)\}$$

and is an **n-ary concept**.

Def (Concept Algebra)

The **n-dimensional concept algebra** of $\mathcal{M}$ is the natural algebra of n-ary concepts

$$CA_n(\mathcal{M}) = \langle \text{n-ary concepts}, \cap, -, c_i \rangle_{i \in \mathbb{N}}$$

c_i – cylindrification, corresponds to $\exists x_i$

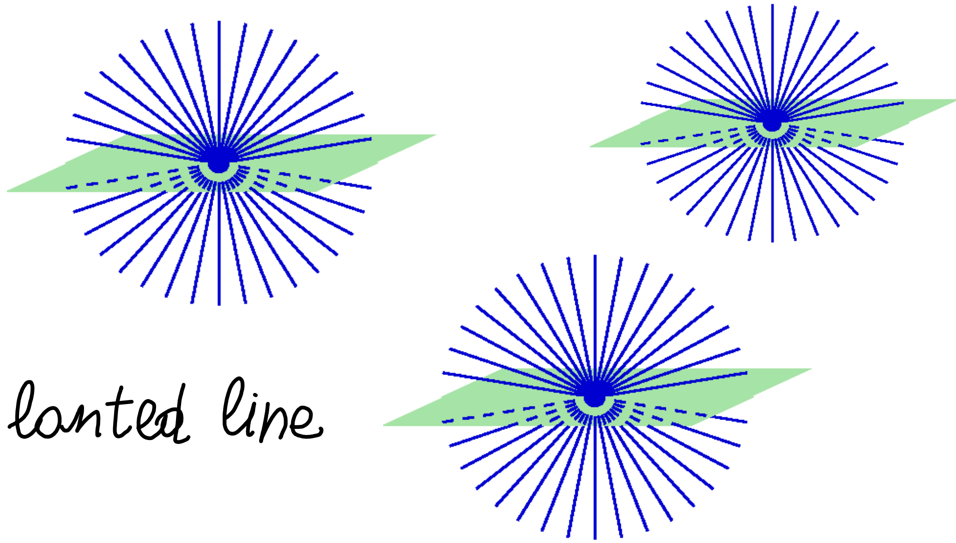
$\cap$ and $-$ correspond to $\wedge$ and $\neg$

$$\langle \mathbb{R}^4, \mathrm{col}^{T}\cdot \rangle \equiv_\Delta \langle \mathbb{R}^4, \tau \rangle^{\cdot} \equiv_\Delta \langle \mathbb{R}^4, \sigma \rangle \equiv_\Delta \langle \mathbb{R}^4, \lambda \rangle$$

$$CA_n \langle \mathbb{R}^4, \mathrm{col}^{T}\cdot \rangle = CA_n \langle \mathbb{R}^4, \tau \rangle^{\cdot} = CA_n \langle \mathbb{R}^4, \sigma \rangle = CA_n \langle \mathbb{R}^4, \lambda \rangle$$

Classical Spacetime CS is the system of non-horisontal lines

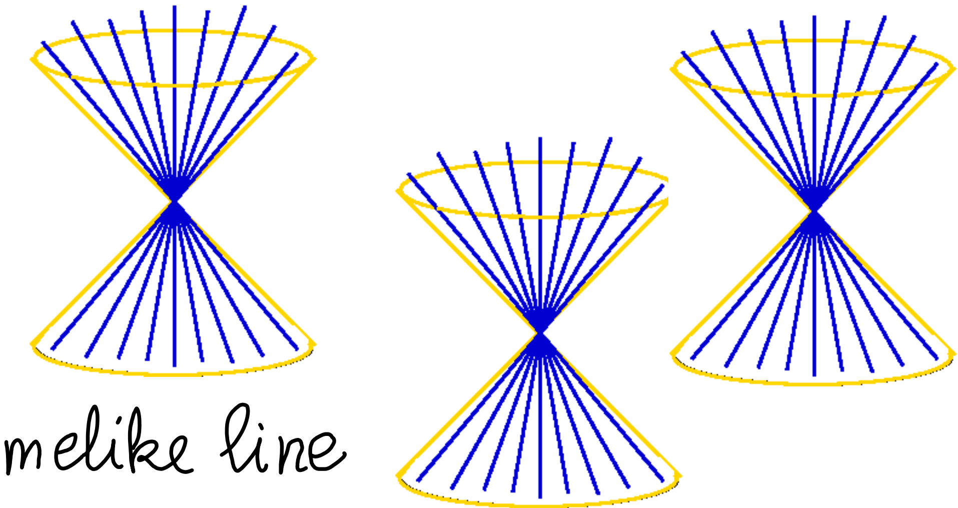
$$CS = \langle \mathbb{R}^4, col^\infty \rangle$$



$col^\infty(a,b,c) \iff$ a, b, c are on a slanted line

Relativistic Spacetime RS is the system of timelike lines

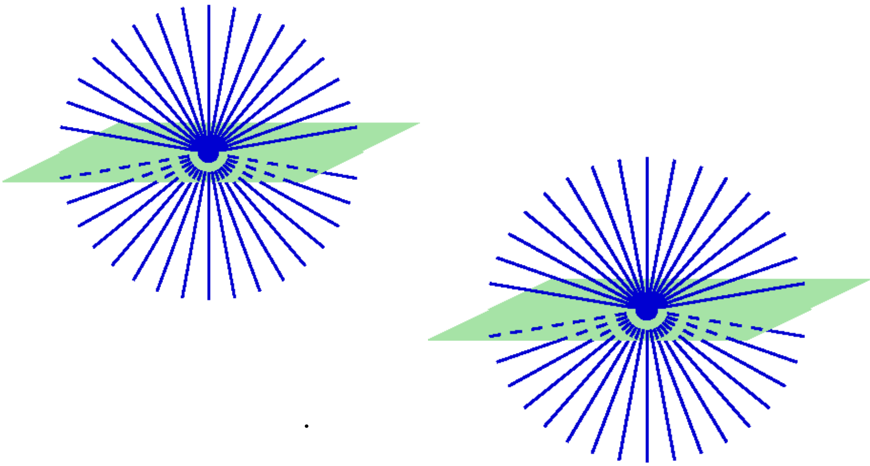
$$RS = \langle \mathbb{R}^4, col^T \rangle$$



$col^T(a,b,c) \iff$ a, b, c are on a timelike line

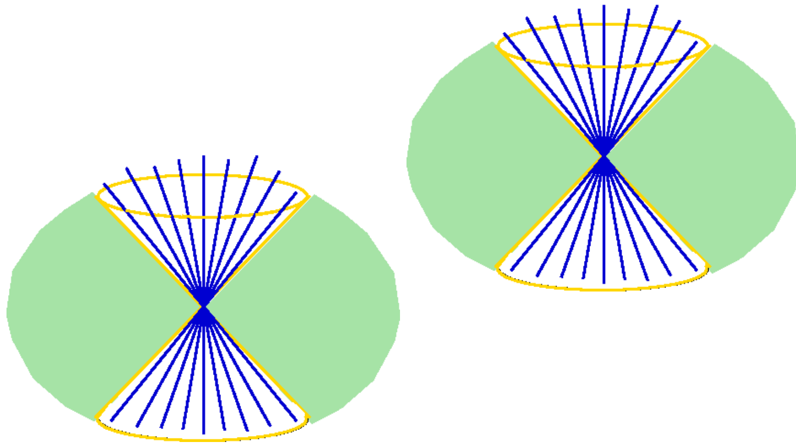
Classical Spacetime

$CS = \langle \mathbb{R}^4, col^\infty \rangle$



Relativistic Spacetime

$RS = \langle \mathbb{R}^4, col^T \rangle$



The green parts form an equivalence relation

green $\sim$ - binary

Transitive closure of the green parts is everything

σ

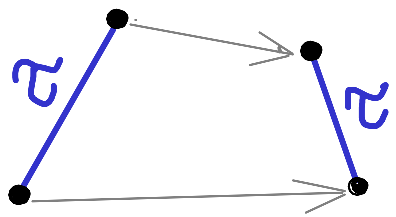
Thm

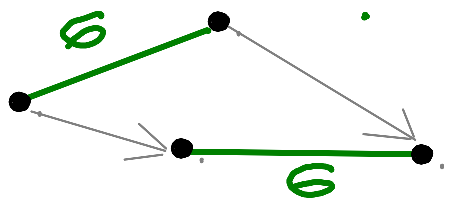
No nontrivial equivalence relation can be defined in RS.

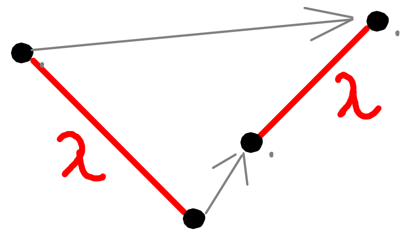
Thm

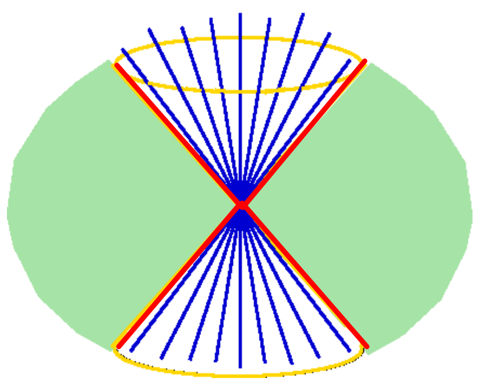
No nontrivial equivalence relation can be defined in $\mathcal{R}S$.

Proof: First we show that any two timelike connected events can be taken to each other by an automorphism of $\mathcal{R}S$, and the same holds for spacelike and lightlike connected events.









Lorentz transformations composed by translations and dilations are automorphisms of $\mathcal{R}S$.

Then show that transitive closures of τ, σ, λ have one block.

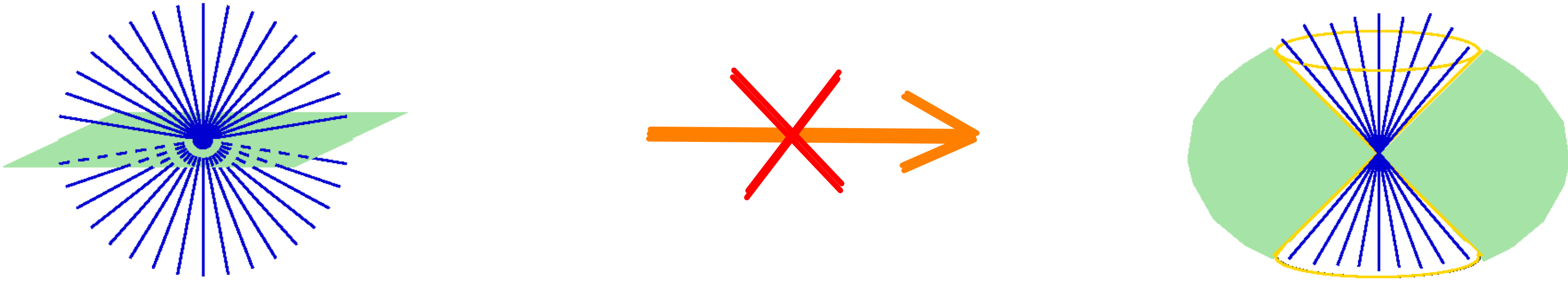
Corollary $\mathcal{R}S \not\equiv_{\Delta} CS$

Classical Spacetime

$CS = \langle \mathbb{R}^4, col^{\infty} \rangle$

Relativistic Spacetime

$RS = \langle \mathbb{R}^4, col^{T} \rangle$



Thm (Corollary)

The theory of CS cannot be interpreted in the theory of RS

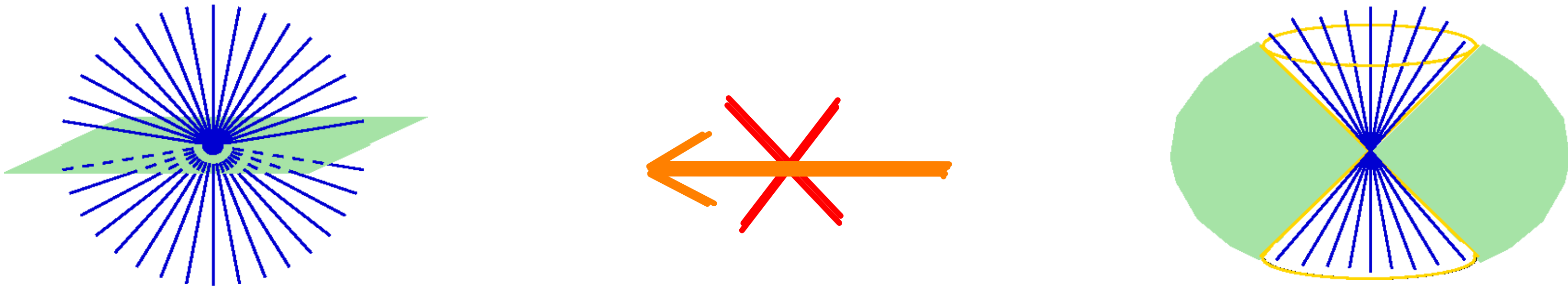
Interpretations are homomorphisms between concept algebras

Classical Spacetime

$CS = \langle \mathbb{R}^4, col^{\infty} \rangle$

Relativistic Spacetime

$RS = \langle \mathbb{R}^4, Col^{T} \rangle$



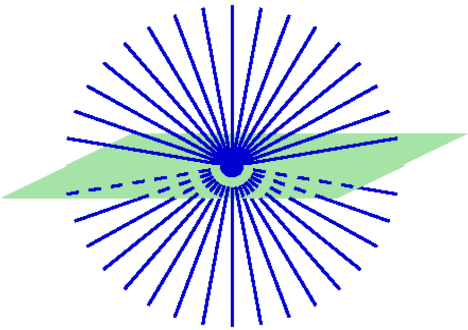
Thm

The theory of RS cannot be interpreted to the theory of CS, either

Reason: RS is conceptually richer than CS (even when CS is enriched with more structure)

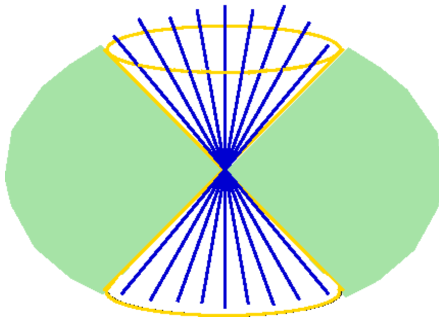
Classical Spacetime

$$\mathcal{CS} = \langle \mathbb{R}^4, col^{\infty} \rangle$$

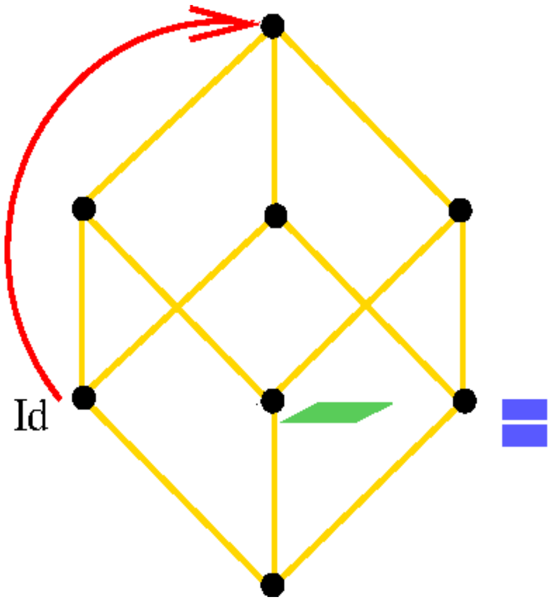


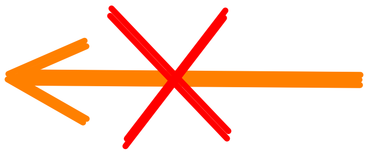
Relativistic Spacetime

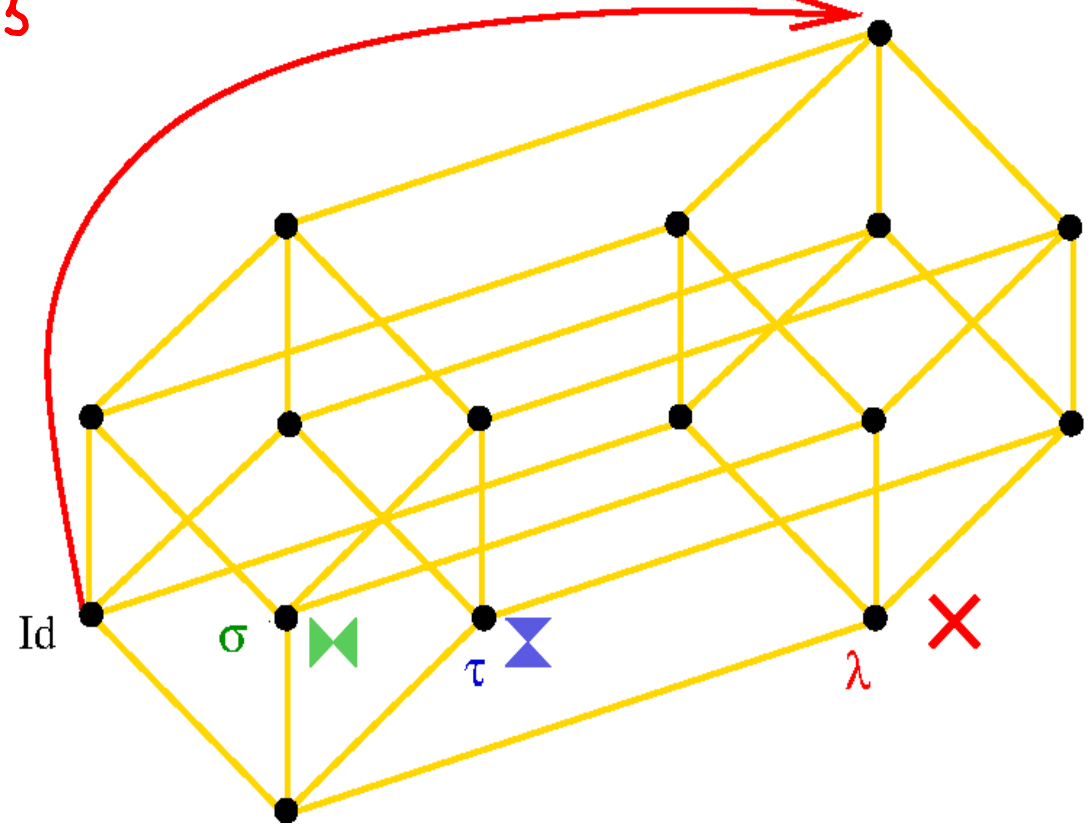
$$\mathcal{RS} = \langle \mathbb{R}^4, col^{T} \rangle$$



2-dim CA-s

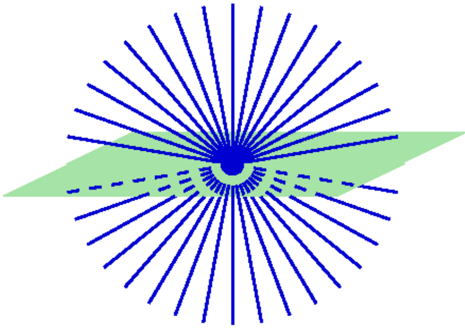




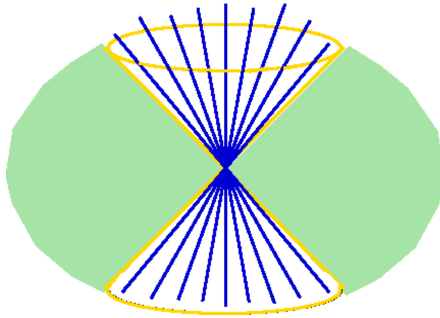


Many Sorted FOL (new entities can be defined)

$$CS \rightleftarrows RS$$



interpretations



Proof : Coordinatsation (Hilbert)

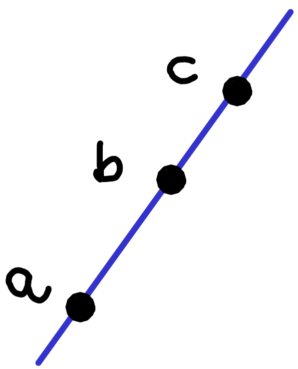
$$CS \neq_{\triangle} RS$$

Proof : AutGroup(CS) $\not\cong$ AutGroup(RS)

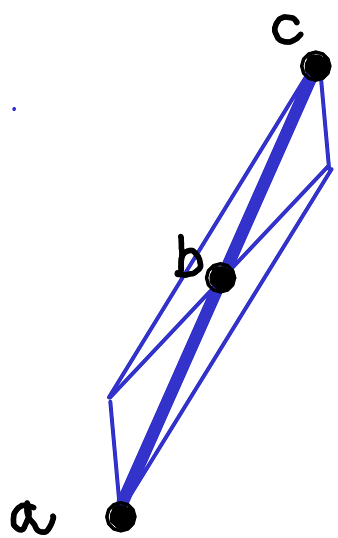
Structures of ternary relations

infinite : rational ratios between timelike collinear points can be defined in both RS an CS

$q \in \mathbb{Q}$



$\varphi_q(a,b,c)$ $\quad col(a,b,c) \wedge$ "$ab : bc = q$"

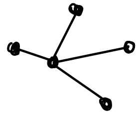


$ab : bc = 1$

Atomic

Describing the atoms:

Col can be defined.

Coordinatisation corresponding to  can be defined

Each algebraic numer can be defined (by its min. pol.)

For every three 4-tuples $p, q, r \in (\text{Algebraic} \cap \mathbb{R})^4$

$\psi_{p,q,r}(a,b,c)$ "there is a coordinatisation according to which the coordinates of a, b, c are p, q, r"

CA_n is atomic

This works for arbitrary real closed field or for $\mathbb{Q}$. What about other O Fields?

Thank you!