When generalised definitional equivalence implies definitional equivalence

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Big question in logical philosophy of science

When should we count two theories as formally equivalent?

Proposals

- Definitional equivalence (D-equivalence)
- Generalised definitional equivalence (GD-equivalence) aka Morita equivalence

Questions

- How important is this generalisation when it comes to logical reconstructions of scientific theories?
- For which kinds of theories does it make a difference?

Overview

Concepts of equivalence

When GD-equivalence implies D-equivalence

Theories

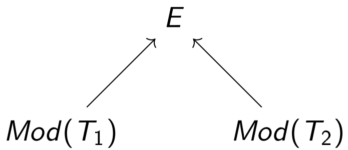
A **theory** T comes with of

- a language of predicate logic, $L(T)$,
- a class of $L(T)$-structures, $Mod(T)$.

Definitional equivalence

Definition

Theories are **D-equivalent** iff their classes of models have a common definitional expansion.



Definitional equivalence

Limitations

Definitional expansions do not allow to introduce

- new sort symbols.
- abstraction or pairing functions ("imaginary elements").

Generalised definitional equivalence

Definition

Theories are **GD-equivalent** iff their classes of models have a common stepwise GD-expansion.

Allows introduction of new sorts together with auxiliary functions:

- subsorts & inclusion functions
- product sorts & projection functions
- coproduct sorts (disjoint unions) & injection functions
- quotient sorts & abstraction functions

Generalised definitional equivalence

Examples

- Versions of affine geometry
  (Barrett & Halvorson, 2016)
- Versions of special relativity
  (Budapest school around Andreka and Nemeti)

D-equivalence and GD-equivalence

Clearly

$$\begin{array}{ccc} \text{Definitional} & \Rightarrow & \text{Generalised definitional} \\ \text{equivalence} & \not\Leftarrow & \text{equivalence} \end{array}$$

D-equivalence and GD-equivalence

Question

"For sufficiently strong single-sorted theories, does Morita equivalence imply definitional equivalence? Of course, the really interesting question here is what 'sufficiently strong' could mean in this context."

(Barrett & Halvorson 2016, p.576)

Overview

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When GD-equivalence implies D-equivalence

Q. "For sufficiently strong single-sorted theories, does generalised definitional equivalence imply definitional equivalence?"
(Barrett & Halvorson, 2016, p.576)

A. Sequential theories with strong elimination of imaginaries are sufficiently strong.

Roughly: theories such that all imaginary elements of their models (i.e. objects that are results of abstraction) can be identified with some of their real elements.

Some helpful concepts

Definition

T has **elimination of imaginaries** iff for every equivalence formula $\phi(\bar{x}, \bar{y})$ of T, there is a formula defining a tuple-valued abstraction function F in any model $\mathscr{A}$ (i.e. $\mathscr{A} \models \phi[\bar{a}_1, \bar{a}_2] \iff F(\bar{a}_1) = F(\bar{a}_2)$).

Theories with elimination of imaginaries are so rich that one can

- represent all imaginary elements over a given model by tuples,
- define abstraction operators for all defb. equivalence relations.

Some helpful concepts

Definition

T has **strong elimination of imaginaries** iff T has elimination of imaginaries and for every $L(T)$-formula of the form $\bar{x} = \bar{y}$ (i.e. $x_1 = y_1 \wedge \ldots \wedge x_n = y_n$), there is an $L(T)$-formula $\varepsilon(\bar{x}, z)$ such that $T \models \forall \bar{y} \exists ! z \forall \bar{x} (\bar{x} = \bar{y} \leftrightarrow \varepsilon(\bar{x}, z))$.

Roughly: tuples of elements can be viewed as individual elements.

Some helpful concepts

Definition

T is **sequential** iff it directly interprets adjunctive set theory:

(1) $\exists x \forall y \; y \notin x$

(2) $\forall x \forall y \exists z \forall u (u \in z \leftrightarrow u \in x \vee u = y)$

Roughly: theories with coding

Examples: ZF, PA, PRA

When GD-equivalence implies D-equivalence

Theorem

Suppose single-sorted theories T_1 and T_2 are

- sequential and
- have strong elimination of imaginaries.

If T_1 and T_2 are GD-equivalent, then T_1 and T_2 are D-equivalent.

Upshot

The criterion of GD-equivalence is ...

... useful for streamlined versions of theories with weak mathematical component.

E.g. Budapest-style axiomatisations of SR.

... not necessary for expressively rich theories (e.g. extending a rich mathematical theory) or having a highly expressive background logic.

E.g. ZFCU + eigenaxioms, real analysis + eigenaxioms

Sketch of a proof

The conceptual completion T^{eq} of T

Add quotient sorts and abstraction functions for all eq. relations.

The syntactic category $\mathbb{T}$ of T

Roughly: a categorical version of Lindenbaum-Tarski idea for FOL.

Sketch of a proof

Important facts

- T_1 GD-equivalent to $T_2 \implies \mathbb{T}_1^{eq} \simeq \mathbb{T}_2^{eq}$.
- T has elimination of imaginaries $\implies \mathbb{T} \simeq \mathbb{T}^{eq}$.

Therefore: T_1 GD-equivalent to $T_2 \implies \mathbb{T}_1 \simeq \mathbb{T}_2$.

Question

If $\mathbb{T}_1 \simeq \mathbb{T}_2$, then how are T_1 and T_2 related?

Sketch of a proof

Conjecture

$\mathbb{T}_1 \simeq \mathbb{T}_2 \implies T_1$ and T_2 bi-interpretable via simple interpretations.

Sketch of a proof

Final steps

- For theories with strong elimination of imaginaries, bi-interpretability via simple interpretations implies bi-interpretability via 1-dimensional, identity-preserving interpretations.
- For sequential theories, bi-interpretability via 1-dimensional, identity-preserving interpretations implies definitional equivalence (Friedman-Visser Theorem, 2014).

Open questions

Main question

Is the central conjecture provable?

Can we simplify the condition? In particular:

Q. Does strong elimination of imaginaries entail sequentiality?

Q. Does sequentiality + elimination of imaginaries entail strong elimination of imaginaries?

The End.

Some helpful concepts

Imaginary elements over a model $\mathscr{A}$:

Equivalence classes of the form $\overline{a}/_\phi$, where $\phi(\overline{x}, \overline{y})$ is a formula such that $\{(\overline{a}, \overline{b}) : \mathscr{A} \models \phi\left[\overline{a}, \overline{b}\right]\}$ is an equivalence relation.

Examples:

- n-tuples,
- rational numbers over $\mathbb{N}$,
- lines and circles over models of elementary geometry.