Several Steps in the Procedure of Hilbert's Axiomatic Method

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Status Quaestionis

- Synthetic (or progressive) method (Cellucci 2000, 2002, 2013).
- Into the meta-theoretical reflection (Majer 2001, 2006, 2014).
- Analytic (or regressive) method (Peckhaus 2002, 2006).

Synthetic (or Progressive) Method

- A synthetic (or progressive) reading of Hilbert's axiomatic method has been sharply defended by Carlo Cellucci (2000, 2002, 2013): i.e., the axiomatic method is the method to derive theorems from axioms already and arbitrarily established which in themselves have no meaning.

Into the Meta-theoretical Reflection

- A reading that places Hilbert's axiomatic method into the meta-theoretical reflection has been advanced by Ulrich Majer (2001, 2006, 2014): i.e., the axiomatic method is the method to verify whether axioms (or systems of axioms) already identified satisfy properties such as completeness, independence, and consistency, and to investigate their mutual relations.

Analytic (or Regressive) Method

- According to this interpretation, the method is an *analytic* (or regressive) *method*: i.e., given a particular scientific field, it is useful for reaching the necessary and sufficient conditions (axioms) of the known results (theorems) of the field and for organizing both in a well-structured and reasonably grounded axiomatic theory (Peckhaus 2002, 2006).

A More Comprehensive View

- According to me, the method must be understood as a very flexible tool of inquiry, and so in order to lead *analytically* (i.e., through the search of the axioms) to an axiomatic, well-structured and reasonably grounded theory – not necessarily a 'formal' one – for a given scientific field, the method must *dynamically* include both *synthetic procedures* and *meta-theoretical reflections* (Formica 2019, Formica – Friend 2021).

Main Sources

- *Axiomatic Thought* (Conference, Zürich 1917)
- *Nature and Mathematical Knowledge* (Lectures, Göttingen 1919-1920)

Step 1: Assembling the Established Facts

"For a more detailed characterization of the axiomatic method, I would like to say the following. If one wants to investigate any field scientifically, first of all, one has to gather the known facts of the discipline" (Hilbert 1919-1920, p. 18)

Step 2: The Choice of Primitive Terms and Pradicate

"When we assemble the facts of a definite, more-or-less comprehensive field of knowledge, we soon notice that these facts are capable of being ordered. This ordering always comes about with the help of a certain *framework of concepts* [*Fachwerk von Begriffen*] in the following way: a concept of this framework corresponds to each individual object of the field of knowledge, and a logical relation between concepts corresponds to every fact within the field of knowledge. The framework of concepts is nothing other than the *theory* of the field of knowledge" (Hilbert 1918, pp. 1107-1108).

Step 3: A First Selection of the Axioms

"If we consider a particular theory more closely, we always see that a few distinguished propositions of the field of knowledge underlie the construction of the framework of concepts, and these propositions then suffice by themselves for the construction, in accordance with logical principles, of the entire framework.

[…]. These fundamental propositions can be regarded from an initial standpoint as the *axioms of the individual field of knowledge*: the progressive development of the individual field of knowledge then lies solely in the further logical construction of the already mentioned framework of concepts" (Hilbert 1918, p. 1108).

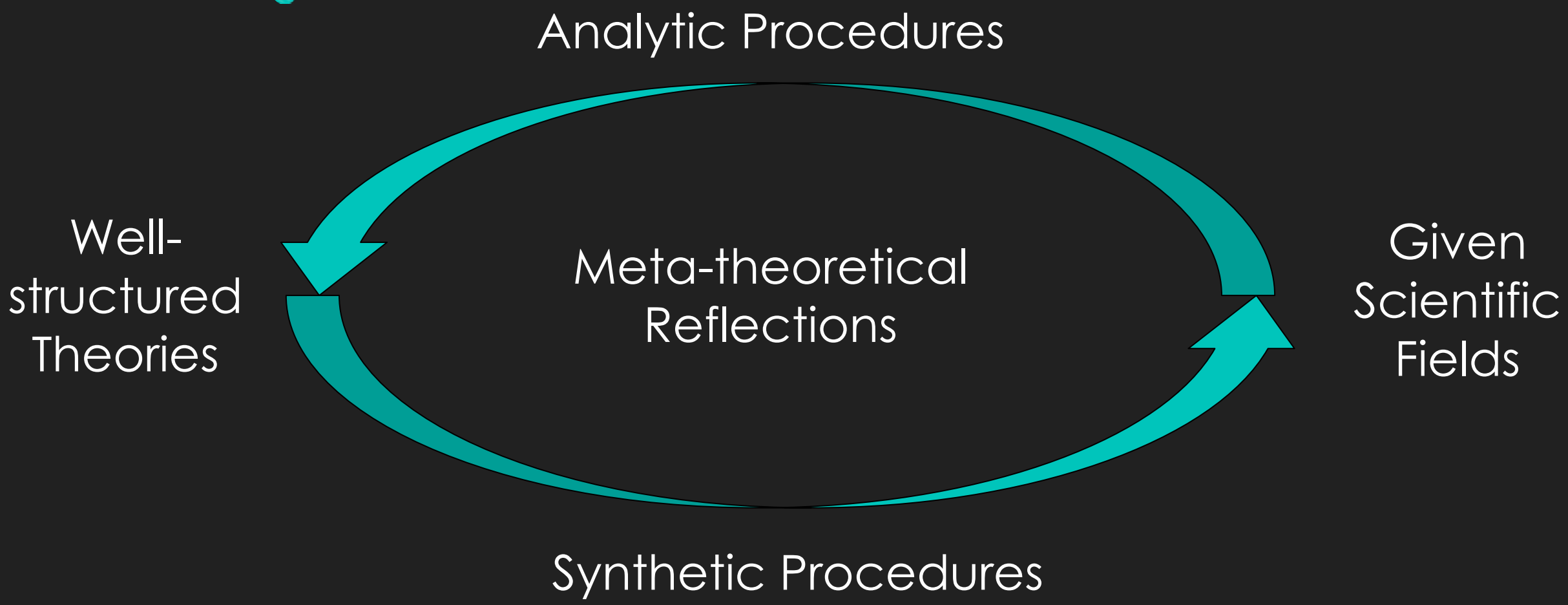
Step 4: Advancing with the Logical Reduction and the Meta-theoretical Inquiry

"Having established such a system of axioms, one can also ask whether the axioms (at least in part) are not themselves provable. This question can be understood in two different ways. It may be a question of deciding whether an axiom can be deduced from the others, namely, whether it is a superfluous axiom. […]. But a proof for the axioms can also be sought by choosing as axioms propositions of a more general character. […]. Progress during proofs of such a kind is made if the foundations of the science are located at a deeper level; in this way, a further step is taken in the logical reduction" (Hilbert 1919-1920, pp. 18-19).

Step 5: Looking for Logically Possible Alternative Theories

"If geometry is to serve as a model for the treatment of physical axioms, we shall try [...] by adjoining new axioms to arrive gradually at more specialised theories. [...]. The mathematician will have also to take account not only those theories coming near to reality, but also, as in geometry, of all logically possible theories" (Hilbert 1900, p. 454).

The Procedure of Hilbert's Axiomatic Method



Five steps to be included

1) Assembling the established facts of the field to be axiomatized.
2) The choice of primitive terms and predicates (i.e., a language) for that field.
3) A first selection of the axioms.
4) Advancing with the logical reduction and the meta-theoretical inquiry.
5) Looking for other logically possible axiomatic theories.

For More Extensive Arguments

- G. FORMICA, *On the Procedural Character of Hilbert's Axiomatic Method*, in *Quaestio*, 19 (2019), pp. 459-482.
- G. FORMICA – M. FRIEND, *In the Footsteps of Hilbert: The Andréka-Németi Group's Logical Foundations of Theories in Physics*, in J. MADARÁSZ – G. SZÉKELY (eds.), *Hajnal Andréka and István Németi on Unity of Science*, Springer, Cham 2021, pp. 383-408.

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- U. MAJER, *Hilbert's Axiomatic Approach to the Foundations of Science – A Failed Research Program?*, in V. HENDRICKS / K.F. JØRGENSEN / J. LÜTZEN / S.A. PEDERSEN (eds.), *Interactions: Mathematics, Physics and Philosophy, 1860-1930*, Springer, Dordrecht 2006, pp. 155-183.
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- V. PECKHAUS, *Predeductive Reasoning*, in C. CELLUCCI / P. PECERE (eds), *Demonstrative and Non-demonstrative Reasoning in Mathematics and Natural Science*, Edizioni dell'Università degli Studi di Cassino, Cassino 2006, pp. 9-25.