On the gap between definitional and categorical equivalence of theories

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Definitional equivalence of FOL theories

$$\mathcal{L}(T_1) \underset{tr_2}{\overset{tr_1}{\rightleftarrows}} \mathcal{L}(T_2)$$

T_1 and T_2 are definitionally equivalent if there are interpretations between them that are each other's inverses.

Categorical equivalence of FOL theories

$\mathcal{Mod}(T)$ is Mod(T) as category. Two versions.

$$\mathcal{Mod}(T_1) \xrightarrow{F} \mathcal{Mod}(T_2)$$

T_1 and T_2 are categorically equivalent if their model categories are equivalent (as categories).

Gap

$$\text{Def.eq.} \quad \underset{\not\longleftarrow}{\longrightarrow} \quad \text{Cat.eq.}$$

Barrett-Halvorson

Hudetz: definable categorical equivalence

Why is this gap interesting?

Whetherall, Erlangen program

This talk:

an example, a theorem and a conjecture on the gap

An Example

$$\text{Def.eq.} \quad \underset{\not\Leftarrow}{\Rightarrow} \quad \text{Cat.eq.}$$

How big is this gap?

More refined picture:

Def. eq. $\overset{\longrightarrow}{\not\longleftarrow}$ GDef. eq. $\overset{\longrightarrow}{\not\longleftarrow}$ Cat. eq.

Laurenz Hudetz

How big is this gap?

We present a set of continuum many theories on a finite language,
each categorically equivalent to the others, but
no two of them Gdef-eq.

This answers a question from a Barrett-Halvorson paper.

The Example

We present a set of continuum many theories on a finite language, each categorically equivalent to the others, but no two of them Gdef-eq.

Language: 0 (constant), suc (unary function), R (unary relation).

$T(S) := \{R(suc^n(0)) : n \text{ in } S\} + \{\neg R(suc^n(0)) : n \text{ not in } S\} + Th(0,suc)$,

where S is a set of natural numbers and Th(0,suc) denotes the theory of natural numbers with 0 and suc as operations.

The Example continued

T(S) := {R(suc^n(0)) : n in S} + {¬ R(suc^n(0)) : n not in S} + Th(0,suc).

A set S of natural numbers is called irregular if it displays all finite patterns. For example, {0,2,4,6,…} is regular, because the pattern (x,suc(x)) does not appear in it. There are continuum many such S.

We show that T(S) and T(Z) are categorically equivalent whenever S and Z are irregular.

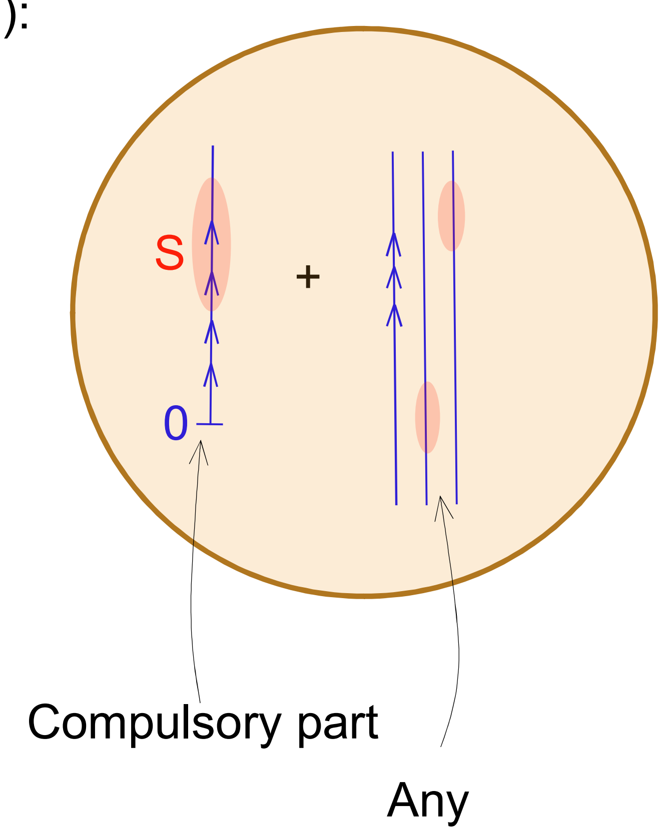
T(S) and T(Z) can be definitionally equivalent but each T(S) can be Gdefinitionally equivalent to only countably many other theories because there are countably many definitions of the basic concepts.

The Example continued

$T(S)$:= {R(suc^n(0)) : n in S} + {¬ R(suc^n(0)) : n not in S} + Th(0,suc).

T(S) and T(Z) are categorically equivalent whenever S, Z are irregular:

One model of T(S):



The Example continued

T(S) and T(Z) are categorically equivalent whenever S, Z are irregular:

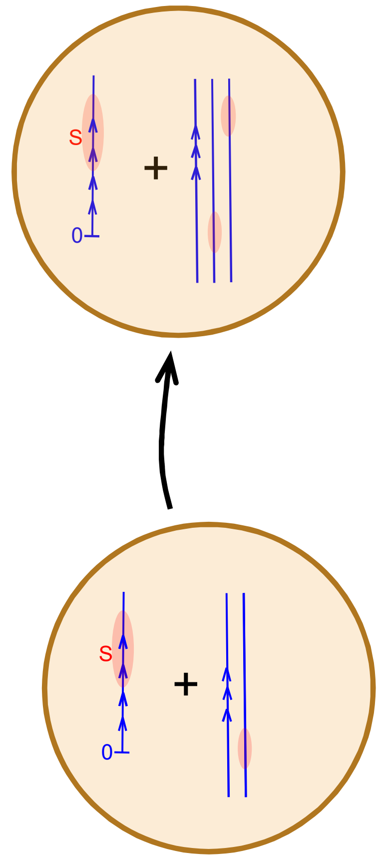
Morphism:

Elementary embedding =

Identity on first part and
any embedding on rest.

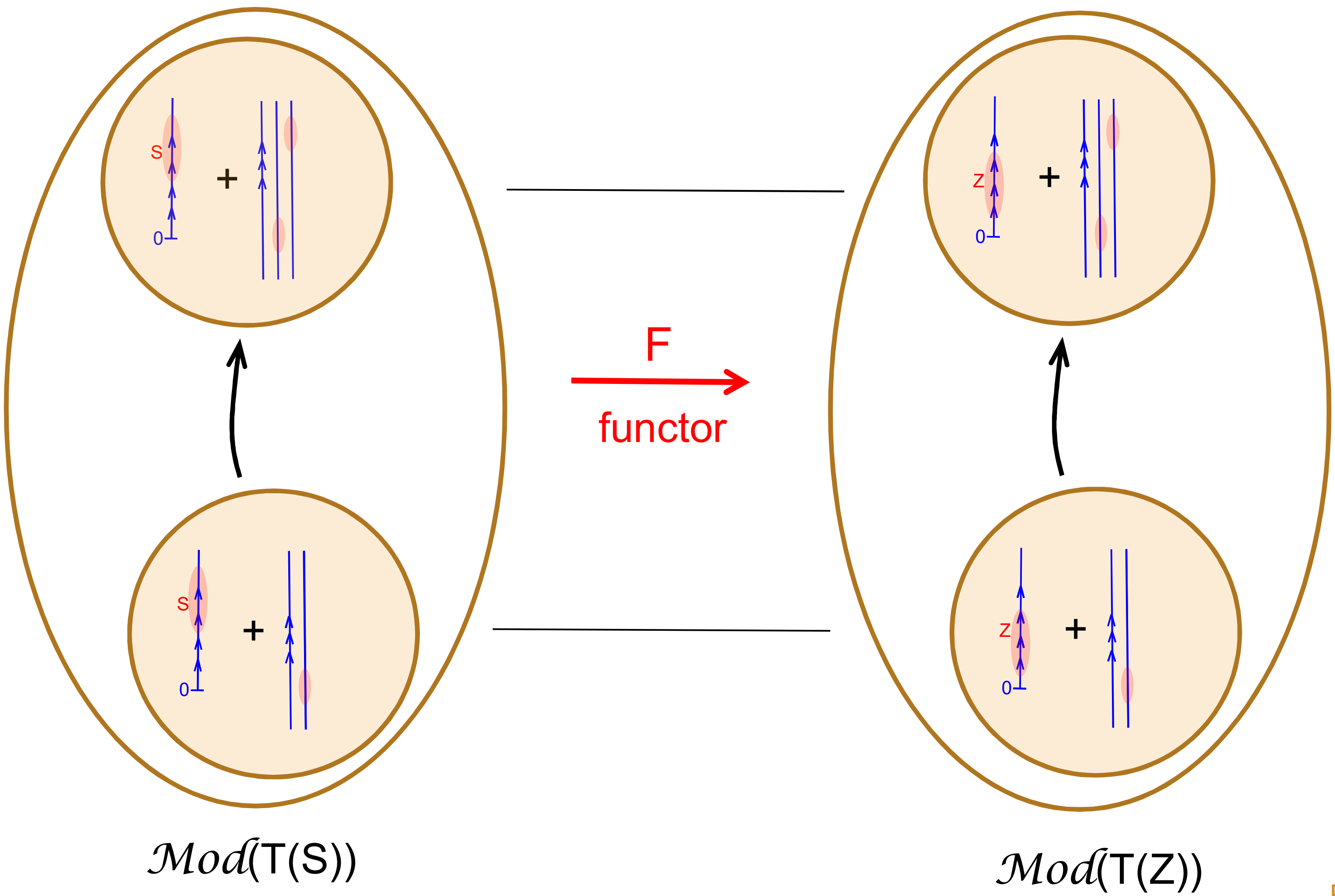
Reason: S is irregular.

Thus S does not play any role in the model category of T(S).



The Example continued

T(S) and T(Z) are categorically equivalent whenever S, Z are irregular:



The Example finished

T(S) and T(Z) are categorically equivalent whenever S, Z are irregular:

You can mix linguistically wildly different but categorycally equivalent theories from the T(S)s. E.g.,

$\{\neg R(0)\}$ and $\{R(0) \rightarrow \varphi : \varphi \text{ in } T(S)\}$

are such if 0 in S and S is irregular.

Never stop at a negative result.

This could have been the motto of Leon Henkin.

Concrete Categorical equivalence of theories

$\mathcal{M}od(\mathrm{T})$ comes with a natural set-structure, “the” forgetful functor to $\mathcal{S}et$.

T_1 and T_2 are called concrete categorically equivalent iff there is a functor F that is an isomorphism between their model categories and commutes with the forgetful functors:

$$\begin{tikzcd}
\mathcal{M}od(\mathrm{T}_1) \arrow[rr, "\mathrm{F}"] \arrow[dr] & & \mathcal{M}od(\mathrm{T}_2) \arrow[dl] \\
& \mathcal{S}et &
\end{tikzcd}$$

A Theorem

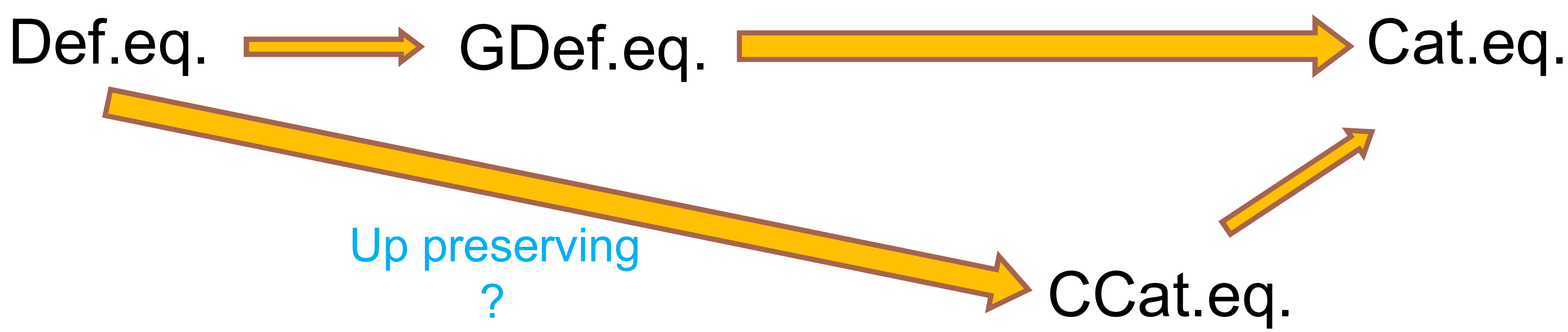
Ultraproducts are model constructions that are characteristic to FOL.

The concrete functors between $\mathcal{Mod}$(T(S)) and $\mathcal{Mod}$(T(Z)) that we constructed preserve ultraproducts up to isomorphisms.

Theorem 1.

If T_1 and T_2 are concrete categorically equivalent by a functor F that preserves ultraproducts, then T_1 and T_2 are definitionally equivalent.

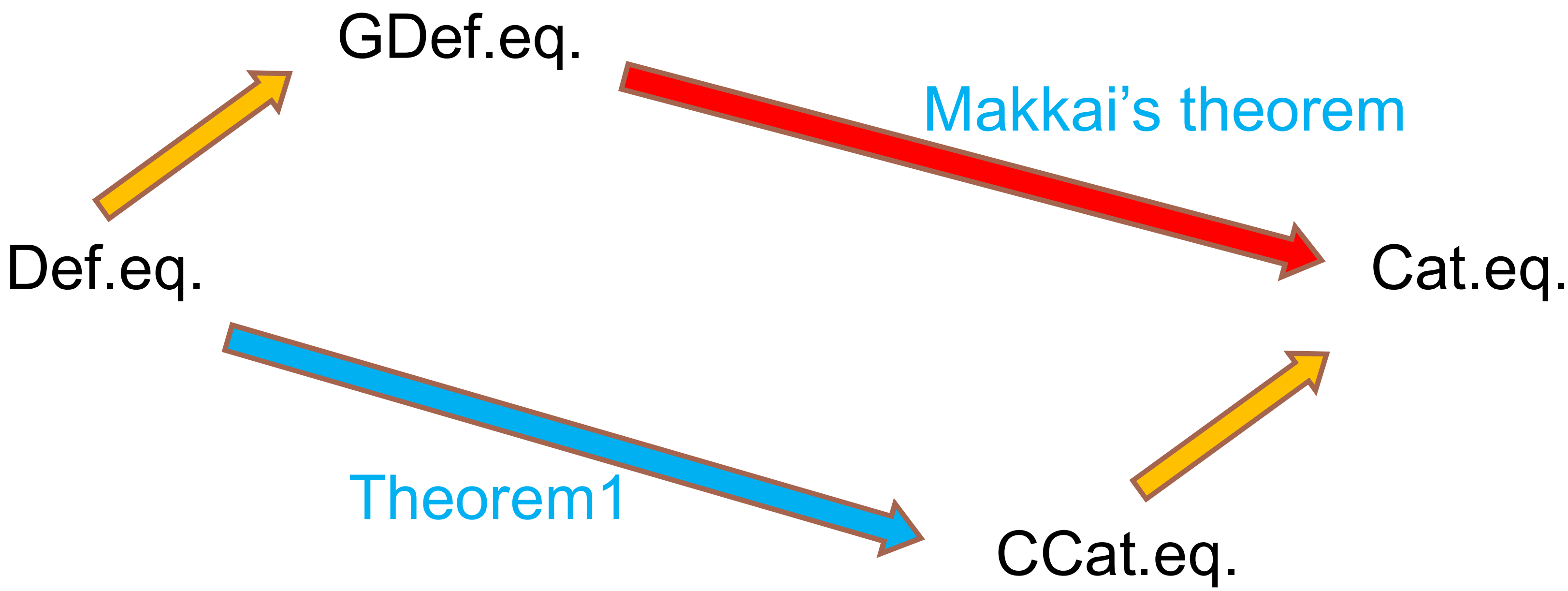
Theorem continued



Theorem 1 answers another question from the same Barrett-Halvorson paper (what properties of F ensure Def.eq.=Cat.eq)

Theorem 1 is an analogon of Makkai's ultracategory theorem.

Question



Is there a deeper analogy between the red and blue arrows?

Conjecture:

Conjecture:
T is concrete categorically equivalent to MinkGeo implies that T is definitionally equivalent to MinkGeo.

It seems that Theorem 1 may be applied to prove the above.

The Conjecture may be true for other theories, too, in place of MinkGeo: EucGeo, some forms of SpecRel, etc.

Related to another question from the same Barrett-Halvorson paper (what theories ensure Def.eq.=Cat.eq). Laurenz Hudetz.

Would give a kind of justification for the Erlangen program!

Thank you for your attention.