Semirings and categories ~~coalgebras and Hopf algebras~~ defined from logical structures

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from logic(ish) input to algebraic(ish) output

- Burnside-Schanuel semiring

$$\text{(logic)} \to \text{(distributive category)} \to \text{(semiring)}$$

- weakly initial objects in the category of models and embeddings of a first-order theory
- coalgebras and Hopf algebras of finite models

distributive category

Definition: Category $\mathcal{C}$ with finite products and coproducts such that the canonical maps

$$\varnothing \to X \times \varnothing$$

$$X \times Y \sqcup X \times Z \to X \times (Y \sqcup Z)$$

are isomorphisms.

Remark Could play with two symmetric monoidal structures, one of which distributes across the other, but details are surprisingly involved.

category of definable sets and functions

Given logic $\mathcal{L}$ in the signature S and S-structure X, consider

objects: $\langle n, A \rangle$ where

$$A = \{\mathbf{x} \in X^n \mid X \models \phi(\mathbf{x}) \text{ for some } \phi \in \mathcal{L}_X\}$$

i.e. $\mathcal{L}$-definable subset of some X^n, with parameters from X

morphism from $\langle n, A \rangle$ to $\langle m, B \rangle$: $\mathcal{L}$-definable subset of X^{n+m} that is the graph of a function from A to B.

These form a category, to be denoted Def(X).

category of formulas and provable functions

Given logic $\mathcal{L}$ and theory T, consider

objects: $\langle \mathbf{x}, \phi \rangle$ where $\phi \in \mathcal{L}$ containing no free variables other than $\mathbf{x}$

morphism from $\langle \mathbf{x}, \phi \rangle$ to $\langle \mathbf{y}, \psi \rangle$ is $\langle \mathbf{x} \sqcup \mathbf{y}, \xi \rangle$ such that

$$T \vdash \forall \mathbf{x} \big(\phi(\mathbf{x}) \to \exists! \mathbf{y} \xi(\mathbf{x}, \mathbf{y}) \big)$$

$$T \vdash \forall \mathbf{x}, \mathbf{y} \big(\phi(\mathbf{x}) \wedge \xi(\mathbf{x}, \mathbf{y}) \to \psi(\mathbf{y}) \big)$$

i.e. T proves that ξ defines the graph of a function from ϕ to ψ.

These form a category, to be denoted Def_T.

distributive categories from logical data

Take $\mathcal{L}$ to be classical first order logic (possibly many-sorted). Then both Def(X) and Def_T

- have terminal objects and pullbacks
  (in the case of Def(X), these are computed as in Set)
- have finite coproducts
- are boolean (subobject lattices are boolean algebras; every subobject is a coproduct summand)
- are distributive.

semirings

All semirings (for us) will be commutative and unital, that is:

A semiring is a set with two commutative, unital binary operations $\circledast$ and $\boxplus$ such that the former distributes over the latter:

$$x \circledast (y \boxplus z) = (x \circledast y) \boxplus (x \circledast z)$$

$$(x \boxplus y) \circledast z = (x \circledast z) \boxplus (y \circledast z)$$

$$x \circledast 0 = 0 \circledast x = 0$$

where 0 is the unit for $\boxplus$.

Grothendieck (semi)ring of a (small) distributive category $\mathcal{C}$

$SK(\mathcal{C})$ is the semiring whose elements are isomorphism classes $[X]$ of objects X, with $[X] \cdot [Y] := [X \times Y]$ and $[X] + [Y] := [X \sqcup Y]$.

$K(\mathcal{C})$, the Grothendieck ring of $\mathcal{C}$, is the abelian group generated by isomorphism classes $[X]$ of objects X, with the relations $[X \sqcup Y] = [X] + [Y]$. Multiplication is induced by $[X] \cdot [Y] = [X \times Y]$.

Remark Schanuel (1990) calls $SK(\mathcal{C})$ the "Burnside rig of $\mathcal{C}$". There is no standard name for this algebra; I will call it the "Burnside-Schanuel semiring" of $\mathcal{C}$.

Remark Any semiring freely generates a ring; adjunction of categories

$$Ring \underset{\text{inclusion}}{\overset{\text{euler}}{\leftrightarrows}} SemiRing$$

$SK(\mathcal{C})$ determines $K(\mathcal{C})$ purely algebraically, since $K(\mathcal{C}) = \text{euler}\big(SK(\mathcal{C})\big)$.

tropical semiring $\mathbb{N}_{-\infty}$

The underlying set is $\mathbb{N} \cup \{-\infty\}$ with

$$\begin{cases} x \circledast y & := x + y \quad (x, y \neq -\infty) \\ x \circledast -\infty & := -\infty \\ x \boxplus y & := \max\{x, y\} \quad (x, y \neq -\infty) \\ x \boxplus -\infty & := x \end{cases}$$

Same structure, written "multiplicatively": formal symbol q ($q > 1$) underlying set $:= \{0, 1, q, q^2, \ldots, q^n, \ldots\}$

$$x \circledast y := x \cdot y \qquad x \boxplus y := \max\{x, y\}$$

$\mathbb{N}_{-\infty}$ is a finitely presentable semiring. The free semiring generated by the singleton X is $\mathbb{N}[X]$ (the semiring of polynomials in the variable X with non-negative integer coefficients under addition and multiplication) and $\mathbb{N}_{-\infty}$ is isomorphic to $\mathbb{N}[X]$ modulo the two relations $1 + 1 = 1$ and $1 + X = X$.

semi-algebraic sets

Definition 1 *Semi-algebraic sets* are the subsets of $\mathbb{R}^n$ first-order definable (with parameters) in the language of $\cdot \quad + \quad > \quad =$

Definition 2 *Semi-algebraic sets* are finite boolean combinations in $\mathbb{R}^n$ of sets of the form

$$\{\mathbf{x} \in \mathbb{R}^n \mid p(\mathbf{x}) > 0\}$$

where p is a polynomial in the variables $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$.

The two definitions are equivalent by the Tarski-Seidenberg theorem (the theory of real closed ordered fields has quantifier elimination; the projection of a semi-algebraic set is semi-algebraic).

SemiAlg

Let *SemiAlg* be $\mathrm{Def}(\mathbb{R}; +, \cdot, >, =)$, the category of semi-algebraic sets and functions.

remark Morphisms of *SemiAlg* need not be continuous!

Objects X, Y of *SemiAlg* are isomorphic iff there is a semi-algebraic pointwise bijection between them.

This is equivalent to X and Y being semi-algebraically topologically equidecomposable: $X = \bigsqcup_{i=1}^{k} X_i$ and $Y = \bigsqcup_{i=1}^{k} Y_i$, with X_i semi-algebraically homeomorphic to Y_i.

so pretty!

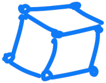
Theorem (Schanuel 1990)

$SK(SemiAlg)$ is a finitely presentable semiring, isomorphic to

$$\mathbb{N}[X] \text{ modulo the relation } X = 2X + 1 \, .$$

in pictures

$X \longleftrightarrow (0,1)$ 

$X^n \longleftrightarrow (0,1)^n$ 

$a_0 + a_1 X + \dots + a_n X^n \longleftrightarrow$ 

use exercise (next slide)

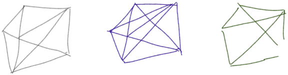
+ Hironaka–Łojasiewicz

elements of $\mathbb{N}[X] \longleftrightarrow$ semi-algebraic sets triangulated with open simplices

$2X + 1 = X + 1 + X = X$

 semialgebraically isomorphic/bijective/to 

this relation induces <u>all</u> equivalences between triangulations (amazing!)



exercise

Show that $(0,1)^n$ (the open n-dimensional cube) and Δ_n^o (the open n-dimensional simplex) are equidecomposable via piecewise linear maps.

There exist semiring homomorphisms (defined purely algebraically)

$$\dim_{alg} : \mathbb{N}[X]/(X = 2X + 1) \to \mathbb{N}_{-\infty}$$

$$\text{euler}_{alg} : \mathbb{N}[X]/(X = 2X + 1) \to \mathbb{Z}$$

The product map

$$\mathbb{N}[X]/(X = 2X + 1) \xrightarrow{\dim_{alg} \times \text{euler}_{alg}} \mathbb{N}_{-\infty} \times \mathbb{Z}$$

is injective.

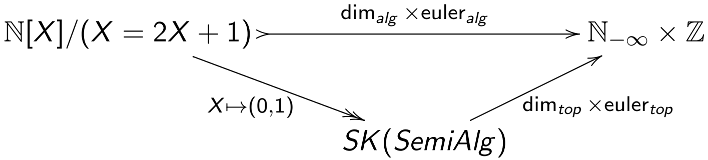
There exist semiring homomorphisms (defined with the help of cell decompositions; deep!)

$$\dim_{top} : SK(SemiAlg) \to \mathbb{N}_{-\infty}$$

$$\text{euler}_{top} : SK(SemiAlg) \to \mathbb{Z}$$

Schanuel's proof

Commutative diagram of semirings



Top horizontal arrow is injective and slanted down arrow is surjective. It follows that the slanted down arrow is an isomorphism.

corollaries of Schanuel's proof

- $K(\mathit{SemiAlg}) = \text{euler}\big(\mathit{SK}(\mathit{SemiAlg})\big) = \mathbb{Z}$
- $\mathit{SK}(\mathit{SemiAlg}) \xrightarrow{\text{dim} \times \text{eu}} \mathbb{N}_{-\infty} \times \mathbb{Z}$ is injective (Schanuel 1990; o-minimally, van den Dries 1998)
- the open interval $(0, 1)$ is the unique (up to isomorphism) semiring generator of $\mathit{SK}(\mathit{SemiAlg})$
- the Burnside-Schanuel semirings of the following distributive categories
  - Def(R) for any real closed field R
  - Def_T where T is the theory of real closed fields
  - $\text{Def}(\mathbb{R}_{exp})$ for any o-minimal expansion of $(\mathbb{R}, \cdot, +, <)$

  are isomorphic to $\mathbb{N}[X]/(X = 2X + 1)$.

moral

- $SK(\mathcal{C})$ carries more information than $K(\mathcal{C})$. $SK(\mathcal{C})$ is never trivial; $K(\mathcal{C})$ is a singleton iff there exists in $\mathcal{C}$ a definable injection from a definable set A to $A - \{a\}$ (where $a \in A$), cf. Krajicek-Scanlon (2000).
- Similar proofs work for the category of semilinear sets, bounded semilinear sets ... Structures whose Burnside-Schanuel semiring is finitely presentable should be thought of as "cellular".
- Question: which finitely presentable semirings arise as $SK(\mathcal{C})$? Elbaz Saban (2020) conjectures that all finitely presentable rings arise as Grothendieck rings of theories.
- Schanuel (1990) adds further relations to the Burnside-Schanuel semiring and suggests that the resulting structure — which is simultaneously a join semilattice and a semiring — can be thought of as an "algebraic approximation" to dimension.

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where is Ramsey theory?

Hilbert's cube theorem

Van der Waerden's theorem

density version

polynomial version

Hindman's theorem

Gallai's theorem

Folkman's theorem

Schur's theorem

Erdos-Rado partition calculus

Ramsey's theorem (finitary)

Ramsey's theorem (infinitary)

metric, structural, topological Ramsey theory

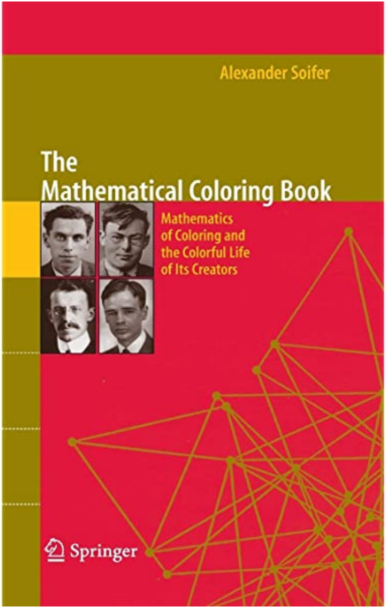
Graham-Rothschild theorem

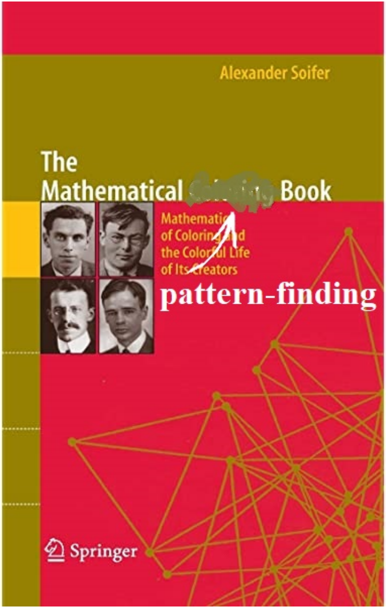
Hales-Jewett theorem

density version

Paris-Harrington theorem

Kanamori-McAloon theorem


Alexander Soifer
The Mathematical Coloring Book
Mathematics of Coloring and the Colorful Life of Its Creators
Springer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very roughly . . .

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  structure $\rightarrow$ model of (first-order) theory
  contains $\rightarrow$ (induced) substructure
  pattern $\rightarrow$ (finite set of) (isomorphism classes of) models

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contains $\rightarrow$ (induced) substructure
pattern $\rightarrow$ (finite set of) (isomorphism classes of) models

- Any large enough model of a theory T will contain a submodel isomorphic to one of a finite set of distinguished models.

very roughly . . .

- Any large enough model of a theory T will contain a submodel isomorphic to one of a finite set of distinguished models.
- The theory T possesses a finite set of countable models such that any infinite model contains a submodel isomorphic to one of them.

set-up

$\mathcal{L}$ relational signature

T first order theory in $\mathcal{L}$

$Mod(T)$ category whose objects are infinite models of T; morphism $f : X \to Y$ is map of underlying sets such that for all $R \in \mathcal{L}$,

$$X \models R(x_1, x_2, \dots, x_n) \text{ iff } Y \models R\big(f(x_1), f(x_2), \dots, f(x_n)\big)$$

A set $\mathcal{I}$ of objects of a category $\mathcal{C}$ is *weakly initial* if

- for all objects $X \in \mathcal{C}$, there exists $I \in \mathcal{I}$ and morphism $I \to X$
- for non-isomorphic $U, V \in \mathcal{I}$ there exists no morphism $U \to V$.

theorem

For every universal theory T in a finite relational signature $\mathcal{L}$ that has an infinite model, the category $Mod(T)$ has a weakly initial set $\mathcal{I}$ of objects.

Weakly initial sets of $Mod(T)$ are finite and unique up to isomorphism (i.e. same cardinality, containing the same isomorphism classes of objects).

theorem, cont'd

If $\mathcal{I}$ is a weakly initial set of objects then

$$1 \leqslant \text{card}(\mathcal{I}) \leqslant 2 \text{ to the power of } \sum_{R \in \mathcal{L}} \mathcal{B}\big(\text{arity}(R)\big)$$

where $\mathcal{B}(n)$ is the n-th *ordered Bell number*, i.e. number of ordered partitions of $\{1, 2, \dots, n\}$

$$\mathcal{B}(n) = \sum_{m=0}^{n} \left\{ {n \atop m} \right\} \cdot m!$$

where $\left\{ {n \atop m} \right\}$ is the Stirling number of the second kind.

theorem, cont'd

For any model X of a universal theory T in a relational signature $\mathcal{L}$ and subset $S \subseteq X$, let $X|_S$ be the $\mathcal{L}$-structure induced on S by restriction from X. Since T is a universal theory, $X|_S \models T$.

Definition A countably infinite model X of T is *self-similar* if for all countably infinite $S \subset X$, $X|_S$ is isomorphic to X (as $\mathcal{L}$-structures).

The weakly initial set of $Mod(T)$ then consists of the (isomorphism types of) countably infinite, self-similar models of T.

example 1

$\mathcal{L}$ consists of binary predicate R; T := tournaments

$$\forall x\ \neg R(x,x)$$
$$\forall x \forall y \big(x \neq y\ \rightarrow\ R(x,y) \triangledown R(y,x)\big)$$

Then $\mathcal{I} = \{I_+, I_-\}$. Both I_+ and I_- have underlying set ω and

$$I_+ = R(i,j) \text{ iff } i < j$$

$$I_- = R(i,j) \text{ iff } i > j$$

example 2

$\mathcal{L}$ consists of k predicates R_i, each of arity n
T := coloring of unordered n-tuples with k colors

$$\forall x_1, x_2, \ldots, x_n \big(R_i(x_1, x_2, \ldots, x_n) \leftrightarrow R_i(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})\big)$$

for each i and for all permutations σ of $1, 2, \ldots, n$

$$\forall x_1, x_2, \ldots, x_n \big(x_i = x_j \rightarrow \neg R_p(x_1, x_2, \ldots, x_n)\big)$$

for each $1 \leqslant i < j \leqslant n$ and $1 \leqslant p \leqslant k$

$$\forall x_1, x_2, \ldots, x_n \Big(\bigwedge_{1 \leqslant i < j \leqslant n} x_i \neq x_j \rightarrow R_1(\mathbf{x}) \vee R_2(\mathbf{x}) \vee \ldots \vee R_k(\mathbf{x})\Big)$$

$$\forall \mathbf{x} \neg \big(R_i(\mathbf{x}) \wedge R_j(\mathbf{x})\big) \quad \text{for all } 1 \leqslant i < j \leqslant k$$

Then $\mathcal{I} = \{I_1, I_2, \ldots I_k\}$. I_j has underlying set ω, with all n-element subsets "colored" R_j.

example 3

$\mathcal{L} = \{<, \prec\}$; $T :=$ $<$ is linear order, $\prec$ is strict partial order.

$\mathcal{I} = \{I_+, I_0, I_-\}$. All have underlying set ω with its $<$.

$$I_+ = i \prec j \text{ iff } i < j$$

$$I_- = i \prec j \text{ iff } i > j$$

$$I_0 = i \prec j \text{ for no } i, j$$

" Any infinite poset contains a countable ascending chain or a countable descending chain or a countable antichain "

proof

- same as the construction of a countable indiscernible sequence (for universal theories in a relational language $\mathcal{L}$, can be done *inside* the model!)
- order-indiscernibles are usually constructed *using* Ramsey's theorem, but the same proof works; use induction on

$$\max\{\text{arity}(R) \mid R \in \mathcal{L}\}$$

- $2^{\sum_{R\in\mathcal{L}} \mathcal{B}(\text{arity}(R))}$ is upper bound on the number of complete quantifier-free types consistent with a universal theory T in the finite relational signature $\mathcal{L}$ (achieved when T is the empty theory)
- Ramsey *basically* constructs indiscernible sequences in his 1928 paper!

Let T be a first-order universal theory in a finite relational language. The category of infinite T-models and embeddings contains a finite weakly initial set of objects, consisting of the countably infinite, self-similar models.

- seems to capture the "natural" level of generality of Ramsey's infinite Ramsey theorem
- implies finite version (which looks quite different)
- extensions to infinite relational languages ("partition calculus")

category-theoretic reformulation?

Let T be a first-order universal theory in a finite relational language. The category of infinite T-models and embeddings contains a finite weakly initial set of objects.

This conclusion holds for first order theories T that do not have a "Ramsey-like" feel!

- fields of characteristic 0 ($\mathbb{Q}$ is strictly initial)
- rings ($\mathbb{Z}$ is strictly initial)
- any theory with an infinite prime model
- G-Sets, for a finite group G

A better way of looking at weakly initial sets of objects would involve bringing in the notion of finitely accessible category, cf. Makkai-Paré (1989) or Adámek-Rosický (1994), and replacing *size of underlying set* by *presentability rank* of the object.

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selected references

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