An instance of Vaught's conjecture using algebraic logic

Tarek Sayed Ahmed

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Tarskian Algebraic Logic–an area interdisciplinary between logic, and algebra (in fact the natural interface between universal algebra and logic) with an accompanying extremely rich geometry that has a varying dimension possibly transfinite–reflected in Tarski?s cylindric algebras now better known as Concept Algebras when applied to (the algebraization of) sophisticated first order theories like spacetime geometries. The canonical examples of the so-called representable algebras, the cylindric set algebra, provide a natural vehicle for Model Theory, since cylindrifications reflect the semantics of existential quantifiers in logic, and are simply forming cylinders that is to say projections in Geometry. Using cylindric set algebras we approach Vaught's conjecture.

In 1961, Robert Vaught asked the following question: Given a complete theory in a countable language, is it the case that it either has countably many or $2^{\aleph_0}$ non-isomorphic countable models? By the number of non-isomorphic countable models is meant the number of their isomorphism-types; that is the number of equivalence classes of countable models w.r.t. the isomorphism relation between structures. We shall just say "the number of countable models" to mean the number of their isomorphism-types.

The positive answer to the question is more commonly know as Vaught's Conjecture. (Vaught's conjecture has the reputation of being the most important open problem in model theory.) However, some logicians do not agree to this sweeping statement.

Quoting Shelah on this: *People say that settling Vaught's conjecture is the* *most important problem in Model theory, because it makes us understand countable models of countable theories, which are the most important models. We disagree with all three statements.*

Morley proved that the number of countable models is either less than or equal to the first uncountable cardinal ($\leq \aleph_1$) or else it has the power of the continuum. This is the best known (general) answer to Vaught's question. Later other logicians confirmed Vaught's conjecture in some special cases of theories, for example:

1. (Shelah)ω-stable theories;

2. (Buechler)superstable theories of finite U-rank;

3. (Mayer)o-minimal theories;

4. (Miller) theories of linear orders with unary predicates;

5. (Steel)theories of trees.

There are also attempts concerning special kinds of models to count and also relations other than isomorphisms between models. Vaught's conjecture can be translated to counting the number of orbits corresponding to the action of S_∞, the symmetric group of ω, on the Polish space of countable models. One way to obtain a positive result is to consider only isomorphisms induced by a *subgroup* G of S_∞ Vaught's conjecture has been confirmed when G is solvable; the best result in this type of investigations, is the case when G is a cli group.

Our work here is inspired by Gabor Sági, who approached Vaught's conjecture using the machinery of algebraic logic.

Cylindric algebras-reflecting both syntax and semantix

Cylindric set algebras are algebras whose elements are relations of a certain pre-assigned arity, endowed with set–theoretic operations that utilize the form of elements of the algebra as sets of sequences. For a set V, $\mathcal{B}(V)$ denotes the Boolean set algebra $\langle \wp(V), \cup, \cap, \sim, \emptyset, V\rangle$. Let U be a set and α an ordinal; α will be the dimension of the algebra. For $X \subseteq {}^{\alpha}U$ and $i, j < \alpha$, let

$$\mathsf{C}_i X = \{s \in {}^{\alpha}U : (\exists t \in X)(t \equiv_i s)\}$$

and

$$\mathsf{D}_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}.$$

The algebra $\langle \mathcal{B}({}^{\alpha}U), \mathsf{C}_i, \mathsf{D}_{ij}\rangle_{i,j<\alpha}$ is called *the full cylindric set algebra of dimension* α with unit (or greatest element) ${}^{\alpha}U$. Any subalgebra of the latter is called a *set algebra of dimension* α. Examples of subalgebras of such set algebras arise naturally from models of first order theories. Indeed, if M is a first order structure in a first order signature L with α many variables, then one manufactures a cylindric set algebra based on M as follows. Let

$$\phi^{\mathsf{M}} = \{s \in {}^{\alpha}\mathsf{M} : \mathsf{M} \models \phi[s]\},$$

(here $\mathsf{M} \models \phi[s]$ means that s satisfies ϕ in M), then the set $\{\phi^{\mathsf{M}} : \phi \in Fm^L\}$ is a cylindric set algebra of dimension α, where Fm^L denotes the set of first order formulas taken in the signature L. To see why, we have:

$$\begin{aligned}
\phi^{\mathsf{M}} \cap \psi^{\mathsf{M}} &= (\phi \wedge \psi)^{\mathsf{M}}, \\
{}^{\alpha}\mathsf{M} \sim \phi^{\mathsf{M}} &= (\neg\phi)^{\mathsf{M}}, \\
\mathsf{C}_i(\phi^{\mathsf{M}}) &= (\exists v_i \phi)^{\mathsf{M}}, \\
\mathsf{D}_{ij} &= (x_i = x_j)^{\mathsf{M}}.
\end{aligned}$$

By Cs_α we denote the class of all subalgebras of full set algebras of dimension α. The (equationally defined) $\mathbf{CA}_\alpha$ class is obtained from cylindric set algebras by a process of abstraction and is defined by a *finite* schema of equations that holds of course in the more concrete set algebras.

Definition .1. Let α be an ordinal. By *a cylindric algebra of dimension* α, briefly a $\mathbf{CA}_\alpha$, we mean an algebra

$$\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, \mathsf{c}_i, \mathsf{d}_{ij} \rangle_{\kappa,\lambda<\alpha}$$

where $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra such that $0, 1$, and d_{ij} are distinguished elements of A (for all $j, i < \alpha$), $-$ and c_i are unary operations on A (for all $i < \alpha$), $+$ and . are binary operations on A, and such that the following equations are satisfied for any $x, y \in A$ and any $i, j, \mu < \alpha$:

(C_1) $\mathsf{c}_i 0 = 0$,

(C_2) $x \leq \mathsf{c}_i x$ $(i.e., x + \mathsf{c}_i x = \mathsf{c}_i x)$,

(C_3) $\mathsf{c}_i(x \cdot \mathsf{c}_i y) = \mathsf{c}_i x \cdot \mathsf{c}_i y$,

(C_4) $\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$,

(C_5) $\mathsf{d}_{ii} = 1$,

(C_6) if $i \neq j, \mu$, then $\mathsf{d}_{j\mu} = \mathsf{c}_i(\mathsf{d}_{ji} \cdot \mathsf{d}_{i\mu})$,

(C_7) if $i \neq j$, then $\mathsf{c}_i(\mathsf{d}_{ij} \cdot x) \cdot \mathsf{c}_i(\mathsf{d}_{ij} \cdot -x) = 0$.

The varieties of representable algebras of dimension α, α an ordinal is defined as via $\mathsf{RCA}_\alpha = \mathbf{SP}\mathsf{Cs}_\alpha$, which turns out to be a variety, that is to say, closed under $\mathbf{H}$, as well.

Let α be an ordinal. An algebra $\mathfrak{A} \in \mathbf{CA}_\alpha$ is *locally finite*, if the dimension set of every element $x \in A$ is finite. The dimension set of x, or Δx for short, is the set $\{i \in \alpha : \mathsf{c}_i x \neq x\}$. Locally finite algebras correspond to Tarski–Lindenbaum algebras of (first order) formulas; in such algebras the dimension set of (an equivalence class of) a formula reflects the number of (finite) set of free variables in this formula. Tarski proved that every locally finite α-dimensional cylindric algebra is representable, i.e. isomorphic to a subdirect product of set algebra each of dimension α. Let Lf_α denote the class of locally finite cylindric algebras.

Let $\mathbf{RCA}_\alpha$ stand for the class of isomorphic copies of subdirect products of set algebras each of dimension α, or briefly, the class of α dimensional representable cylindric algebras. Then Tarski's theorem reads $\mathbf{Lf}_\alpha \subseteq \mathbf{RCA}_\alpha$. This representation theorem is non-trivial; in fact it is equivalent to Gödel's celebrated Completeness Theorem. Completeness in the general case is a huge subject that has provoked extensive research. A natural generalization of Lf_α is Dc_α when α is infinite; $\mathfrak{A} \in \mathsf{Dc}_\alpha$ iff $\alpha \sim \Delta x$ is infinite for all $x \in A$.

Part 1: Counting models omiting types for quantifier logics with infinitely many variables

Morley's result extended to counting models omitting types

Let us first talk about Omitting types for the so-called rich languages where there are infinitely many variables outside each (atomic) formula

We define certain cardinals; it is consistent that such cardinal are uncountable. Throughout this talk we do not assume the continuum hypothesis.

Definition .2. 1. A subset $X \subseteq \mathbb{R}$ is *meager* if it is a countable union of nowhere dense sets. Let covK be the least cardinal κ such that $\mathbb{R}$ can be covered by κ many nowhere dense sets. Let $\mathfrak{p}$ be the least cardinal κ such that there are κ many meager sets of $\mathbb{R}$ whose union is not meager.

2. A *Polish space* is a topological space that is metrizable by a complete separable metric.

Examples of Polish spaces are $\mathbb{R}$, the Cantor set ${}^{\omega}2$ and the Baire space ${}^{\omega}\omega$. These are called *real spaces* because they are Baire isomorphic. Any second countable compact Hausdorff space, like the Stone space of a countable Boolean algebra, is a Polish space (a complete separable metric space).

Theorem .3. *1. The cardinals* covK *and* $\mathfrak{p}$ *are uncountable cardinals, such that* $\mathfrak{p} \leq \mathsf{covK} \leq 2^{\omega}$.

2. The cardinal covK *is the least cardinal such the Baire category theorem* *for Polish spaces fails, and it is also the largest for which Martin's axiom for countable Boolean algebras holds.*

3. *If X is a Polish space, then it cannot be covered by $< \mathsf{covK}$ many meager sets. If $\lambda < \mathfrak{p}$, and $(A_i : i < \lambda)$ is a family of meager subsets of X, then $\bigcup_{i\in\lambda} A_i$ is meager.*

Both cardinals covK and $\mathfrak{p}$ have an extensive literature. It is consistent that $\omega < \mathfrak{p} < \mathsf{covK} \leq 2^\omega$ so that the two cardinals are generally different, but it is also consistent that they are equal; equality holds for example in the Cohen real model of Solovay and Cohen. In this case, Martin's axiom implies that they are both equal to the continuum. Let $\mathfrak{A}$ be any Boolean algebra. The set of ultrafilters of $\mathfrak{A}$ is denoted by $\mathfrak{U}(\mathfrak{A})$. The

Stone topology makes $\mathfrak{U}(\mathfrak{A})$ a compact Hausdorff space. We denote this space by $\mathfrak{A}^*$. Recall that the Stone topology has as its basic open sets the sets $\{N_x : x \in A\}$ where

$$N_x = \{F \in \mathfrak{U}(\mathfrak{A}) : x \in F\}.$$

Let $x \in A$, $Y \subseteq A$ and suppose that $x = \sum Y$. We say that an ultrafilter $F \in \mathfrak{U}(\mathfrak{A})$ *preserves* $Y \iff$ whenever $x \in F$, then $y \in F$ for some $y \in Y$. Now let $\mathfrak{A} \in \mathsf{Lf}_\omega$. For each $i \in \omega$ and $x \in A$ let

$$\mathfrak{U}_{i,x} = \{F \in \mathfrak{U}(\mathfrak{A}) : F \text{ preserves } \{\mathsf{s}^i_j x : j \in \omega\}\}.$$

Then

$$\begin{aligned}\mathfrak{U}_{i,x} &= \{F \in \mathfrak{U}(\mathfrak{A}) : \mathsf{c}_i x \in F \Rightarrow (\exists j \in \omega)\mathsf{s}^i_j x \in F\}\\ &= N_{-\mathsf{c}_i x} \cup \bigcup_{j<\omega} N_{\mathsf{s}^i_j x}.\end{aligned}$$

Let

$$\mathcal{H}(\mathfrak{A}) = \bigcap_{i\in\omega, x\in A} \mathfrak{U}_{i,x}(\mathfrak{A}) \cap \bigcap_{i\neq j} N_{-\mathsf{d}_{ij}}.$$

It is clear that $\mathcal{H}(\mathfrak{A})$ is a G_δ set in $\mathfrak{A}^*$. For $F \in \mathfrak{U}(\mathfrak{A})$, let

$$rep_F(x) = \{\tau \in {}^\omega\omega : \mathsf{s}_\tau^{\mathfrak{A}} x \in F\},$$

for all $x \in A$. Here for $\tau \in {}^\omega\omega$, $\mathsf{s}_\tau^{\mathfrak{A}} x$ by definition is $\mathsf{s}_{\tau\restriction\Delta x}^{\mathfrak{A}} x$. The latter is well defined because $|\Delta x| < \omega$. When $a \in F$, then rep_F is a representation of $\mathfrak{A}$ such that $rep_F(a) \neq 0$. The following theorem establishes a one to one correspondence between representations of locally finite cylindric algebras and Henkin ultrafilters. $\mathsf{Cs}_\omega^{\mathsf{reg}}$ denotes the class of *regular set algebras*; a a set algebra with top element ${}^\alpha U$ is such, if whenever $f, g \in {}^\alpha U$, $f \restriction \Delta x = g \restriction \Delta x$, and $f \in X$ then $g \in X$. This reflects the metalogical property that if two assignments agree on the free variables occurring in a formula then both satisfy the formula or none does.

Theorem .4. *(Gabor Sagi) If $F \in \mathcal{H}(\mathfrak{A})$, then rep_F is a homomorphism from $\mathfrak{A}$*

onto an element of $\mathsf{Lf}_\omega \cap \mathsf{Cs}_\omega^{\mathsf{reg}}$ *with base* ω. *Conversely, if* h *is a homomorphism from* $\mathfrak{A}$ *onto an element of* $\mathsf{Lf}_\omega \cap \mathsf{Cs}_\omega^{\mathsf{reg}}$ *with base* ω*, then there is a unique* $F \in \mathcal{H}(\mathfrak{A})$ *such that* $h = rep_F$.

The next lemma is due to Shelah, and will be used to show that in certain cases uncountably many non-principal types can be omitted.

Lemma .5. *Suppose that* T *is a theory,* $|T| = \lambda$*,* λ *regular, then there exist models* $\mathfrak{M}_i : i < {}^{\lambda}2$*, each of cardinality* λ*, such that if* $i(1) \neq i(2) < \chi$*,* $\bar{a}_{i(l)} \in M_{i(l)}$*,* $l = 1, 2,$*,* $\mathsf{tp}(\bar{a}_{l(1)}) = \mathsf{tp}(\bar{a}_{l(2)})$*, then there are* $p_i \subseteq \mathsf{tp}(\bar{a}_{l(i)})$*,* $|p_i| < \lambda$ *and* $p_i \vdash \mathsf{tp}(\bar{a}_{l(i)})$ *(*$\mathsf{tp}(\bar{a})$ *denotes the complete type realized by the tuple* $\bar{a}$*).*

We shall use the algebraic counterpart of the following corollary obtained by restricting Shelah's theorem to the countable case:

Corollary .6. *For any countable theory, there is a family of* $< {}^{\omega}2$ *countable models that overlap only on principal types.*

Theorem .7. *Assume that* $\kappa < \mathfrak{p}$. *Let* α *be a countable infinite ordinal.*

1. *Let* $\mathfrak{A} \in \mathsf{Dc}_\alpha$ *be countable. Let* $(\Gamma_i : i \in \kappa)$ *be a set of non-principal types in* $\mathfrak{A}$. *Then there is a weak set algebra* $\mathfrak{B}$, *that is,* $\mathfrak{B}$ *has top element a weak space, and a homomorphism* $f : \mathfrak{A} \to \mathfrak{B}$ *such that for all* $i \in \kappa$, $\bigcap_{x \in X_i} f(x) = \emptyset$, *and* $f(a) \neq 0$. *If* $\mathfrak{A}$ *is simple, then* $\mathfrak{p}$ *can be replaced by* covK.

2. *If* $\mathfrak{A} \in \mathsf{Lf}_\alpha$, *and* $(\Gamma_i : i \in \kappa)$ *is a family of finitary non-principal types*

then there is a topological set algebra $\mathfrak{B}$, that is, $\mathfrak{B}$ has top element a Cartesian square, and $\mathfrak{B} \in \mathsf{Cs}_{\alpha}^{reg} \cap \mathsf{Lf}_{\alpha}$ together with a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $\bigcap_{x \in X_i} f(x) = \emptyset$, and $f(a) \neq 0$. If the family of given types are ultrafilters then $\mathfrak{p}$ can be replaced by 2^{ω}, so that $< 2^{\omega}$ types can be omitted.

Proof. For the first part, we have

$$(\forall j < \alpha)(\forall x \in A)(\mathsf{c}_j x = \sum_{i \in \alpha \smallsetminus \Delta x} \mathsf{s}_i^j x.) \tag{1}$$

Now let V be the weak space ${}^{\omega}\omega^{(Id)} = \{s \in {}^{\omega}\omega : |\{i \in \omega : s_i \neq i\}| < \omega\}$. For each $\tau \in V$ for each $i \in \kappa$, let

$$X_{i,\tau} = \{\mathsf{s}_{\tau} x : x \in X_i\}.$$

Here s_{τ} is the unary operation as defined corresponding to τ. For each $\tau \in V$, s_{τ} is a complete Boolean endomorphism on $\mathfrak{A}$ by It thus follows that

$$(\forall \tau \in V)(\forall i \in \kappa) \prod^{\mathfrak{A}} X_{i,\tau} = 0 \qquad (2)$$

Let S be the Stone space of the Boolean part of $\mathfrak{A}$, and for $x \in \mathfrak{A}$, let N_x denote the clopen set consisting of all Boolean ultrafilters that contain x. Then from (1) and (2) it follows that for $x \in \mathfrak{A}$, $j < \beta$, $i < \kappa$ and $\tau \in V$, the sets

$$\mathbf{G}_{j,x} = N_{\mathsf{c}_j x} \setminus \bigcup_{i \notin \Delta x} N_{\mathsf{s}_i^j x} \text{ and } \mathbf{H}_{i,\tau} = \bigcap_{x \in X_i} N_{\mathsf{s}_{\bar{\tau}} x}$$

are closed nowhere dense sets in S. Also each $\mathbf{H}_{i,\tau}$ is closed and nowhere dense. Let

$$\mathbf{G} = \bigcup_{j \in \beta} \bigcup_{x \in B} \mathbf{G}_{j,x} \text{ and } \mathbf{H} = \bigcup_{i \in \kappa} \bigcup_{\tau \in V} \mathbf{H}_{i,\tau}.$$

By properties of $\mathfrak{p}$, $\mathbf{H}$ can be reduced to a countable collection of nowhere dense sets. By the Baire Category theorem for compact Hausdorff spaces, we get that $\mathfrak{H}(\mathfrak{A}) = S \sim \mathbf{H} \cup \mathbf{G}$ is dense in S. Let F be an ultrafilter in $N_a \cap X$. By the very choice of F, it follows that $a \in F$ and we have the following.

$$\begin{aligned}&(\forall j < \beta)(\forall x \in B)(\mathsf{c}_j x \in F \\ &\implies (\exists j \notin \Delta x)\mathsf{s}^i_j x \in F.)\end{aligned} \tag{3}$$

and

$$(\forall i < \kappa)(\forall \tau \in V)(\exists x \in X_i)\mathsf{s}_\tau x \notin F. \tag{4}$$

Let $V = {}^\omega\omega^{Id)}$ and let W be the quotient of V as defined above. That is $W = V/\bar{E}$ where $\tau\bar{E}\sigma$ if $\mathsf{d}_{\tau(i),\sigma(i)} \in F$ for all $i \in \omega$. Define f by $f(x) = \{\bar{\tau} \in W : \mathsf{s}_\tau x \in F\}$, for $x \in \mathfrak{A}$. Then f is a homomorphism such that $f(a) \neq 0$ and it can be easily checked that $\bigcap f(X_i) = \emptyset$ for all $i \in \kappa$, hence the desired conclusion. If $\mathfrak{A}$ is simple, then by the properties of covK, $\mathfrak{H}(\mathfrak{A}) = S \sim \mathbf{H} \cup \mathbf{G}$ is non-empty. Let $F \in H(\mathfrak{A})$ and let $a \in F$. The representation built using such F as above, call it f, has $f(a) \neq 0$, By simplicity of $\mathfrak{A}$, f is an injection, because $kerf = \{0\}$, since $a \notin kerf$ and by simplicity, either $kerf = \{0\}$ or $kerf = \mathfrak{A}$.

2. One proceeds exactly like in the previous item, but using, as indicated above, the fact that the operations s_τ for *any* $\tau \in {}^\omega\omega$ which are definable in locally finite algebras, via $\mathsf{s}_\tau x = \mathsf{s}_{\tau \upharpoonright \Delta x} x$, for any $x \in A$. Furthermore, $\mathsf{s}_\tau \upharpoonright \mathfrak{Nr}_n \mathfrak{A}$ is a complete Boolean endomorphism, so that we guarantee that infimums are preserved and the sets $\mathbf{H}_{i,\tau} = \bigcap_{x \in X_i} N_{\mathsf{s}_{\bar{\tau}} x}$ remain no-where dense in the Stone topology. Now for the second part. Let $\mathfrak{A} \in \mathsf{Lf}_\alpha$, $\lambda < 2^\omega$ and $\mathbf{F} = (X_i : i < \lambda)$ be a family of maximal non-principal finitary types, so that for each $i < \lambda$, there exists $n \in \omega$ such that $X_i \subseteq \mathfrak{Nr}_n \mathfrak{A}$, and $\prod X_i = 0$; that is X_i is a Boolean ultrafilter in $\mathfrak{Nr}_n \mathfrak{A}$. Then by Theorem .5, or

rather its direct algebraic counterpart, there are ${}^{\omega}2$ representations such that if X is an ultrafilter in $\mathfrak{Nr}_n\mathfrak{A}$ (some $n \in \omega$)) that is realized in two such representations, then X is necessarily principal. That is there exist a family of countable locally finite set algebras, each with countable base, call it $(\mathfrak{B}_{j_i} : i < 2^{\omega})$, and isomorphisms $f_i : \mathfrak{A} \to \mathfrak{B}_{j_i}$ such that if X is an ultrafilter in $\mathfrak{Nr}_n\mathfrak{A}$, for which there exists distinct $k, l \in 2^{\omega}$ with $\bigcap f_l(X) \neq \emptyset$ and $\bigcap f_j(X) \neq \emptyset$, then X is principal, so that from Shelah's lemmasuch representations overlap only on maximal principal types. Then there exists a family $(F_i : i < 2^{\omega})$ of Henkin ultrafilters such that $f_i = h_{F_i}$, and we can assume that h_{F_i} is an $\mathbf{CA}_{\alpha}$ isomorphism as follows. Denote F_i by G. Assume, for contradiction, that there is no representation (model) that omits $\mathbf{F}$. Then for all $i < 2^{\omega}$, there exists

F such that F is realized in $\mathfrak{B}_{j_i}$. Let $\psi : 2^\omega \to \wp(\mathbf{F})$, be defined by $\psi(i) = \{F : F \text{ is realized in } \mathfrak{B}_{j_i}\}$. Then for all $i < 2^\omega$, $\psi(i) \neq \emptyset$. Furthermore, for $i \neq k$, $\psi(i) \cap \psi(k) = \emptyset$, for if $F \in \psi(i) \cap \psi(k)$ then it will be realized in $\mathfrak{B}_{j_i}$ and $\mathfrak{B}_{j_k}$, and so it will be principal. This implies that $|\mathbf{F}| = 2^\omega$ which is impossible.

$\square$

Given an equivalence relation there are theorems that assert that either the quotient space is 'small' or else it contains a copy of a specific 'large' set. Two dichotomies showing this tendency are known.

- The Silver Vaught Dichotomy asserts that there are either countably many equivalence classes or there is a perfect set of pairwise inequivalent elements. For any continous action by a Polish group G on a Polish space X, the orbit equivalence relation is conjectured to satisfy the Silver Vaught Dichotomy. This conjecture both implies and is motivated by Vaught's conjecture. In Vaught's conjecture is the particular case. when the group is the symmetric group of permutations on X, and the set X, is the set of non isomorphic models of a theory with domain ω. The relation E is just the equivalence relation of isomorphism. In our case X was a G_δ subset of the Stone space of a countable cylindric algebra.

- Another Dichotomy, called the Glimm Effros dchotomy for an equivalence relation E asserts that E contains a copy of the Vitali equivalence relation E_0 (equivalently there exists a non atomic ergodic measure for E) or else there is a countable Borel separating family for E. This dichotomy originates with Glimm and Effros and is motivated by questions about operator algebra. Glimm proved that the orbit space of a Polish group G action satisfies the Glimm Effros Dichotomy if G is locally compact. Effros proved it for any Polish group G, provded that the equivalence relation is F_σ. There exists S_∞ spaces which violate the Glimm Effros dichotomy, but for the Silver Vaught Dichotomy this is still an open question. In all cases we consider, the Glimm Effros dichotomy implies the Silver Vaught dichotomy.

Theorem .8. *Let $G \subseteq S_\infty$ be a cli group, and let E_G denote the corresponding orbit equivalence relation. Then $|\mathcal{H}(\mathfrak{A})/E_G| \leq \omega$ or $|\mathcal{H}(\mathfrak{A})/E_G| = 2^\omega$*

Proof. It is known that the number of orbits of E_G satisfies the so-called Glimm-Effros Dichotomy. By known results in the literature on the topological version of Vaught's conjecture, we have $\mathcal{H}(\mathfrak{A})/E_G$ is either at most countable or $\mathcal{H}(\mathfrak{A})/E_G$ contains continuum many non equivalent elements (i.e non-isomorphic models). ∎

It is known that the number of orbits of $E = E_{S_\infty}$ *does not* satisfy the Glimm Effros Dichotomy. We note that cli groups cover all natural extensions of abelian groups, like nilpotent and solvable groups. Now we give a topological condition that implies Vaught's conjecture. Let everything be as above with G denoting a Polish subgroup of S_∞. Give $\mathcal{H}(\mathfrak{A})/E_G$ the quotient topology and let $\pi : \mathcal{H}(\mathfrak{A}) \to \mathcal{H}(\mathfrak{A})/E_G$ be the projection map. π of course depends on G, we sometimes denote it by π_G to emphasize the dependence.

Lemma .9. *π is open.*

Proof. To show that π is open it is enough to show for arbitrary $a \in \mathfrak{A}$ that $\pi^{-1}(\pi(N_a))$ is open. For,

$$\pi^{-1}(\pi(N_a)) = \{F \in \mathcal{H}(\mathfrak{A}) : (\exists F' \in N_a)$$
$$(F, F') \in E\}$$
$$= \{F \in \mathcal{H}(\mathfrak{A}) : (\exists F' \in N_a)(\exists \rho \in G)$$
$$s^+_\rho F' = F\}$$
$$= \{F \in \mathcal{H}(\mathfrak{A}) : (\exists F' \in N_a)(\exists \rho \in G)$$
$$F' = s^+_{\rho^{-1}} F\}$$
$$= \{F \in \mathcal{H}(\mathfrak{A}) : (\exists \rho \in G) s^+_{\rho^{-1}} F \in N_a\}$$
$$= \{F \in \mathcal{H}(\mathfrak{A}) : (\exists \rho \in G) a \in s^+_{\rho^{-1}} F\}$$
$$= \{F \in \mathcal{H}(\mathfrak{A}) : (\exists \rho \in G) s^+_\rho a \in F\}$$
$$= \bigcup_{\rho \in G} N_{s^+_\rho a}$$

$\square$

Theorem .10. *If π is closed, then Vaught's conjecture holds.*

Proof. We have $\mathcal{H}(\mathfrak{A})$ is Borel subset of $\mathfrak{A}^*$, the Stone space of $\mathfrak{A}$, and $\mathcal{H}(\mathfrak{A})/E_G$ is a continuous image of $\mathcal{H}(\mathfrak{A})$. Because π is open, $\mathcal{H}(\mathfrak{A})/E_G$ is second countable. Now, since $\mathcal{H}(\mathfrak{A})$ is metrizable, it is normal. Since π is closed, open, continuous, and surjective, so $H(\mathfrak{A})/E_G$ is also normal, hence regular. Thus $\mathcal{H}(\mathfrak{A})/E_G$ can be embedded in $\mathbb{R}^\omega$ (like in the proof of Urysohn's metrization Theorem). If $\mathcal{H}(\mathfrak{A})/E_G$ is uncountable, then being analytic (the continuous image under a map between two Polish spaces of a Borel set), it has the power of the continuum. ∎

Unfortunately, π can't be closed when $G = S_\infty$ (or G sufficiently large as we shall see) and $\mathfrak{A}$ is simple (this is the case when our theory T is complete). Indeed, if it was closed, then as has just been shown, $\mathcal{H}(\mathfrak{A})/E$ is Haussdorf.

A well known fact says that: when the quotient map is open, $\mathcal{H}(\mathfrak{A})/E$ is Hausdorf iff E is closed. We show that when $\mathfrak{A}$ is simple, then E is not closed. For, assume towards a contradiction that E is closed, that is $\sim E$ is open. Let $(F, F') \notin E$. Then for some $a \in F$, $b \in F'$, $N_a \times N_b \cap E = \emptyset$, i.e., for all $\tau \in S_\infty$, $a.s_\tau^{+} b = 0$. This last situation is of course impossible because one can choose τ so that $\Delta a \cap \Delta s_\tau^{+} b = \emptyset$. Here we we used the fact that when $\mathfrak{A}$ is simple and $\Delta x \cap \Delta y = \emptyset$, then $x.y \neq 0$.

An algebraic proof to Morley's theorem endowed with OTT

We next give a new proof of Morley's theorem; we also count the number of models omitting a given family of types.

Theorem .11. *Suppose T is a first order complete theory in a countable language with equality.*

1. (Morley) *If T has more than ω_1 countable models, then it has 2^ω countable models. The same statement holds for theories not necessarily complete, in countable languages with or without equality.*

2. *If $(\Gamma_i : i < \omega)$ be a family of non-isolated types, then the number of*

non isomorphic countable models, omitting this family, is either ω, ω_1 *or* ${}^{\omega}2$

Proof. Let T be a first order theory in a countable language with equality, and let $\mathfrak{A} = \mathfrak{Fm}_T$. Then S_∞ is a Polish group with respect to composition of functions and the topology it inherits from the Baire space ${}^{\omega}\omega$. Consider the map $J : \mathsf{S}_\infty \times \mathcal{H}(\mathfrak{A}) \longrightarrow \mathcal{H}(\mathfrak{A})$ defined by $J(\rho, F) = \mathsf{s}_\rho^+ F$ for all $\rho \in \mathsf{S}_\infty$, $F \in \mathcal{H}(\mathfrak{A})$. Then J is a well defined action of S_∞ on $\mathcal{H}(\mathfrak{A})$. Also J is a continuous map from $\mathsf{S}_\infty \times \mathcal{H}(\mathfrak{A})$ (with the product topology) to $\mathcal{H}(\mathfrak{A})$ because for an arbitrary $a \in A$,

$$J^{-1}(N_a \cap \mathcal{H}(\mathfrak{A})) = \bigcup_{\tau \in \mathsf{S}_\infty} \Big(\{\mu^{-1} : \mu \in \mathsf{S}_\infty,\ \mu|_{\Delta a}$$

$$= \tau|_{\Delta a}\} \times [N_{\mathsf{s}_\tau^+ a} \cap \mathcal{H}(\mathfrak{A})].$$

It follows that the the orbit equivalence relation is analytic. By Burgess' Theorem if there are more than ω_1 orbits, then there are 2^ω orbits. But the number of orbits here is exactly the number of non-isomorphic countably infinite models of T. This completes the proof. For the part on omittung types, set $\mathcal{H}_{omit} = \mathcal{H}(\mathfrak{Fm}_T)\cap\bigcap_{i\in\omega,\tau\in W}\bigcup_{\varphi\in\Gamma_i} N_{-s_\tau^+(\varphi/\equiv_T)}$, where $W = \{\tau \in {}^\omega\omega : |i : \tau(i) \neq i| < \omega\}$. Clearly, $l\mathcal{H}_{omit}$ is G_δ, so it is Polish. For the remaining part one uses locally finite QA_ωs instead of Lf_ωs. $\square$

Example .12. (i) Let T be a countable theory. Then the number of non isomorphic models is equal to the number of models omitting a given a set of $< \lambda$ many types are the same
$\iff |\mathcal{H}(\mathfrak{Fm}_T)| > |\bigcup_{i\in\lambda,\tau\in W}\bigcap_{\varphi\in\Gamma_i} N_{\mathsf{s}_\tau^+(\varphi/\equiv_T)}|$.
The next example shows that this may fail to happen: Consider non standard models of arithmetic. $\mathbb{N}$ is an atomic model, which means that the neat n-reduct of the locally finite cylindric algebra $\mathfrak{Fm}_T$ based on $T = \mathsf{Th}(\mathbb{N})$ is atomic

for each n. For each $n \in \omega$, let Γ_n be the set of co-atoms in the neat n-reduct. These are non-principal types and a model M omits them $\iff$ it is atomic, hence it is isomorphic to $\mathbb{N}$ because atomic models are unique. But Peano arithmetic is unstable, so it has ${}^{\omega}2$ many non-isomorphic countable models (non-standard models of arithmetic). Another example exhibiting the same phenomena: Let T be the theory of algebraically closed fields of characterstic zero. Then T is ω stable and it has countably many non-isomorphic models; for each $\alpha \leq \omega$, there is a model of transcendence degree α over the rationals. Take the types as above. Iin this case the all subalgebras of the n-neat reducts are atomic. Then the the field of algebraic number is the only countable model omitting this family of types. This is an atomic model. This theory has also another countable ω-saturated model, which is that of transcendence degee ω.

(ii) There is a somewhat amusing Theorem of Vaught's that says that a countable theory cannot have exactly two models. We show that this is not the case when we require that the constructed odels omit a given family of non-principal types. Take the language $L = \{c_n : n \in \omega\}$. Then a model M of T is determined by how many extra elements it has, i.e by $|\{b \in \mathsf{M} : b \neq c_n^{\mathsf{M}}\}|$. So T is ω_1 categorial and since T has only infinite models it is complete. Also T has countably many non isomorphic models, M_α with α extra elements for $\alpha \leq \omega$. Consider the m type $\Gamma = \bigwedge_{i \neq j < m}\{v_i \neq v_j\} \cup \{v_0 \neq c_n : n \in \omega\} \cup \{v_1 \neq c_n : n \in \omega\} \ldots \{v_{m-1} \neq c_n : n \in \omega\}$. Then Γ is non-principal and it is omitted by exactly m models namely $\mathsf{M}_0, \mathsf{M}_1, \ldots \mathsf{M}_{m-1}$. This can be generalized for complete strongly minimal theories which have countable models of dimension α, $\alpha \leq \omega$.

(iii) We show that there is a theory having exactly ω_1 models omitting continuum many types. Take the first order countable theory in the language $\{<, c_0, c_1, \ldots\}$ where $<$ is a binary relation symbols and the c_i''s $(i \leq \omega)$ are constants. Let T be the L theory which states that $<$ is a linear order and that $c_i \neq c_j$ for $i \neq j$. Take $\Gamma_1 = \{v_1 \neq c_i : i \in \omega\}$ and for every injective $f \in {}^{\omega}\omega$, let $\Gamma_f = \{c_{f(i)} > c_{f(i+1)} : i \in \omega\}$. Consider the set of non-principal types $\mathcal{G} = \{\Gamma_1, \Gamma_f : f \in {}^{\omega}\omega\}$. Then a model M omits $\mathcal{G} \iff$ it is a countable well order. The family $\mathcal{G}$ is uncountable. Making this family countable would violate Vaught's conjecture in $L_{\omega_1,\omega}$. Indded let

T be a countable theory and $\{\Gamma_i : i < \omega\}$ be a family of non-principal types omitted by exactly ω_1 models. Then the $L_{\omega_1,\omega}$ sentence $\bigwedge T \wedge \bigwedge_{n\in\omega}(\neg(\exists \bar{v_n}) \bigwedge_{\phi\in\Gamma_n} \phi(\bar{v_n}))$ violates Vaught's conjecture; for it has ω_1 countable models.

Vaught's conjecture holding for distinguishible ordinary models and pairwise non-isomorphic models

Distinguishable models

We define an equivalence relation on ultrafilters that turns out to be Borel. This implies that it satisfies the Glimm-Effros dichotomy, and so has either countably many or else continuum many equivalence classes. The equivalence relation we introduce corresponds to a non-trivial equivalence relation between models which is weaker than isomorphism and stronger than elementary equivalence.

Definition .13 (Notation)**.** Let $\mathcal{F}$ be an ultrafilter of a locally finite (cylindric or quasi-polyadic) algebra $\mathfrak{A}$. For $a \in A$ define

$$Sat_{\mathcal{F}}(a) = \{t|_{\Delta a} : t \in {}^{\omega}\omega,\ s_t^{+} a \in \mathcal{F}\}.$$

Throughout, $\mathfrak{A}$ is countable. We define an equivalence relation $\mathcal{E}$ on the space $\mathcal{H}(\mathfrak{A}))$ that turns out to be Borel.

Definition .14. Let $\mathcal{E}$ be the following equivalence relation on $\mathcal{H}(\mathfrak{A})$:

$$\mathcal{E} = \{(\mathcal{F}_0, \mathcal{F}_1) : (\forall a \in A)(|Sat_{\mathcal{F}_0}(a)| = |Sat_{\mathcal{F}_1}(a)|)\}.$$

We say that $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{H}(\mathfrak{A})$ are *distinguishable* if $(\mathcal{F}_0, \mathcal{F}_1) \notin \mathcal{E}$. We also say that two models of a theory T are distinguishable if their corresponding ultrafilters in $\mathcal{H}(CA(T) = \mathsf{Fm}_T)$ are distinguishable. That is, two models are distinguishable if they disagree in the number of realizations they have for some formula. Then $\mathcal{E}$ is Borel in the product space $\mathcal{H}(\mathfrak{A}) \times \mathcal{H}(\mathfrak{A})$.

If X be a Polish space and E a Borel equivalence relation on X. We call E

smooth if there is a Borel map f from X to the Cantor space ${}^{\omega}2$ such that

$$xEy \Leftrightarrow f(x) = f(y).$$

Note that E is smooth iff E admits a *countable Borel separating family*, i.e., a family (A_n) of Borel sets such that

$$xEy \Leftrightarrow \forall n(x \in A_n \leftrightarrow y \in A_n).$$

Clearly, if E is smooth then it is Borel (but the converse is not true). A standard example of a non-smooth Borel equivalence relation is the following: On $2^{\mathbb{N}}$, let E_0 be defined by

$$xE_0y \Leftrightarrow \exists n \forall m \geq n(x(m) = y(m)).$$

We say that the equivalence relation E, on a Polish space X, satisfies the *Glimm-Effros Dichotomy* if either it is smooth or else it contains a copy of E_0. Clearly, for an equivalence relation E, E satisfies the Glimm-Effros Dichotomy implies that E satisfies the *Silver-Vaught Dichotomy*, that is, E has either countably many classes or else perfectly many classes (X has a perfect subset of non-equivalent elements).

Theorem .15 (Harrington-Kechris-Louveau)**.** *Let X be a Polish space and E a Borel equivalence relation on X. Then E satisfies the Glimm-Effros Dichotomy.*

It follows directly from this theorem, replacing X with $\mathcal{H}(\mathfrak{A})$ that $\mathcal{E}$ satisfies the Glimm-Effros dichotomy and so has either countably many equivalence classes or else perfectly many.

Corollary .16. *Let T be a first order theory in a countable language (with or without equality). If T has an uncountable set of countable models that are pairwise distinguishable, then actually it has such a set of size $2^{\aleph_0}$.*

Vaught's conjecture holds when counting weak models in rich languages

A rich language is one for which outside any atomic formula thet are infinitely many variables–and the rest is like first order logic. Recall that rich languages (corresponding to Dc_α) enjoy an omitting types theorem; for $< \mathfrak{p}$ many non-principal types, and the types can contain infinitely many variables (unlike first order logic). However, the models that omit a countable set of non-principal types is only a weak model, and it can be proved that there are cases, where it has to be a weak model.

Example .17. Let T be the theory of dense linear order without endpoints. Then T is complete. Let $\Gamma(x_0, x_1 \ldots)$ be the set

$$\{x_1 < x_0, x_2 < x_1, x_3 < x_2 \ldots\}.$$

(Here there is no bound on free variables.) A model $\mathfrak{M}$ omits Γ if and only if $\mathfrak{M}$ is a well ordering. But T has no well ordered models, so no model of T omits Γ. However T locally omits Γ because if $\phi(x_0, \ldots x_{n-1})$ is consistent with T, then $\phi \wedge \neg x_{n+2} < x_{n+1}$ is consistent with T. Note that Γ can be omitted in a weak model.

But first a some definitions

Definition .18. Let $\mathfrak{A}$ and $\mathfrak{B}$ be set algebras with bases U and W respectively. Then $\mathfrak{A}$ and $\mathfrak{B}$ are *base isomorphic* if there exists a bijection $f : U \to W$ such that $\bar{f} : \mathfrak{A} \to \mathfrak{B}$ defined by $\bar{f}(X) = \{y \in {}^{\alpha}W : f^{-1} \circ y \in x\}$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$

Definition .19. An algebra $\mathfrak{A}$ is *hereditary atomic*, if each of its subalgebras is atomic.

Finite Boolean algebras are hereditary atomic of course, but there are infinite hereditary atomic Boolean algebras; any Boolean algebra generated by by its atoms is hereditary atomic, for example the finite co-finite algebra on any set. An algebra that is infinite and complete is not hereditary atomic, whether atomic or not.

Example .20. Hereditary atomic algebras arise naturally as the Tarski-Lindenbaum algebras of certain countable first order theories, that abound. If T is a countable complete first order theory which has an an ω-saturated model, then for each $n \in \omega$, the Tarski-Lindenbuam Boolean algebra $\mathfrak{Fm}_n/T$ is hereditary atomic. Here $\mathfrak{Fm}_n$ is the set of formulas using only n variables. For example $Th(\mathbb{Q}, <)$ is such with $\mathbb{Q}$ the ω saturated model.

A well known model-theoretic result is that T has an ω saturated model iff

T has countably many n types for all n. Algebraically n-types are just ultrafilters in $\mathfrak{Fm}_n/T$. And indeed, what characterizes hereditary atomic algebras is that the base of their Stone space, that is the set of all ultrafilters, is at most countable.

Lemma .21. *Let $\mathfrak{B}$ be a countable Boolean algebra. If $\mathfrak{B}$ is hereditary atomic then the number of ultrafilters is at most countable; of course they are finite if $\mathfrak{B}$ is finite. If $\mathfrak{B}$ is not hereditary atomic the it has 2^{ω} ultrafilters.*

Our next theorem is the natural extension of Vaught's theorem to variable rich languages. However, we address only languages with finitely many relation symbols. (Our algebras are finitely generated, and being simple, this is equivalent to that it is generated by a single element.)

Theorem .22. *Let $\mathfrak{A} \in Dc_\alpha$ be countable simple and finitely generated. Then the number of non-base isomorphic representations of $\mathfrak{A}$ is 2^ω.*

Proof. Let $V = {}^\alpha\alpha^{(Id)}$ and let $\mathfrak{A}$ be as in the hypothesis. Then $\mathfrak{A}$ cannot be atomic, least hereditary atomic. By .21, it has 2^ω ultrafilters.

For an ultrafilter F, let $h_F(a) = \{\tau \in V : s_\tau a \in F\}$, $a \in \mathfrak{A}$. Then $h_F \neq 0$, indeed $Id \in h_F(a)$ for any $a \in F$, hence h_F is an injection, by simplicity of $\mathfrak{A}$. Now $h_F : \mathfrak{A} \to \wp(V)$; all the h_F's have the same target algebra. We claim that $h_F(\mathfrak{A})$ is base isomorphic to $h_G(\mathfrak{A})$ iff there exists a finite bijection $\sigma \in V$ such that $s_\sigma F = G$. We set out to confirm our claim. Let $\sigma : \alpha \to \alpha$ be a finite bijection such that $s_\sigma F = G$. Define $\Psi : h_F(\mathfrak{A}) \to \wp(V)$ by $\Psi(X) = \{\tau \in V :$

$\sigma^{-1} \circ \tau \in X\}$. Then, by definition, Ψ is a base isomorphism. We show that $\Psi(h_F(a)) = h_G(a)$ for all $a \in \mathfrak{A}$. Let $a \in A$. Let $X = \{\tau \in V : s_\tau a \in F\}$. Let $Z = \Psi(X)$. Then

$$Z = \{\tau \in V : \sigma^{-1} \circ \tau \in X\}$$
$$= \{\tau \in V : s_{\sigma^{-1} \circ \tau}(a) \in F\}$$
$$= \{\tau \in V : s_\tau a \in s_\sigma F\}$$
$$= \{\tau \in V : s_\tau a \in G\}.$$
$$= h_G(a)$$

Conversely, assume that $\bar{\sigma}$ establishes a base isomorphism between $h_F(\mathfrak{A})$ and $h_G(\mathfrak{A})$. Then $\bar{\sigma} \circ h_F = h_G$. We show that if $a \in F$, then $s_\sigma a \in G$. Let $a \in F$, and let $X = h_F(a)$. Then, we have

$$\sigma \bar{\circ} h_F(a) = \sigma(X)$$
$$= \{y \in V : \sigma^{-1} \circ y \in h_F(X)\}$$
$$= \{y \in V : s_{\sigma^{-1} \circ y} a \in F\}$$
$$= h_G(a)$$

Now we have $h_G(a) = \{y \in V : s_y a \in$

$G\}$. But $a \in F$. Hence $\sigma^{-1} \in h_G(a)$ so $s_{\sigma^{-1}}a \in G$, and hence $a \in s_\sigma G$.

Define the equivalence relation $\sim$ on the set of ultrafilters by $F \sim G$, if there exists a finite permutation σ such that $F = s_\sigma G$. Then any equivalence class is countable, and so we have ${}^{\omega}2$ many classes, which correspond to the non base isomorphic representations of $\mathfrak{A}$.

$\square$

Theorem .23. *Let T be a countable theory in a rich language, with only finitely many relation symbols, and $\Gamma = \{\Gamma_i : i \in \mathfrak{p}\}$ be non isolated types. Then T has 2^ω weak models that omit Γ. If T is complete we can replace $\mathfrak{p}$ by* covK.

Part 2: Omiting types theorems (OTTs) for finite variable fragments–both positive and negative results

We recall that a class K of Boolean algebras with operators (BAOs) is *atom–canonical* if whenever $\mathfrak{A} \in$ K is atomic and completey additive, then its completion, namely, the complex algebra of its atom structure, in symbols $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, is also in K. This subtle construction may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive BAOs. One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), blows it up, by splitting one or more of its atoms each to infinitely many subatoms, obtaining an (infinite) *countable* atomic $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{Bb}(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{Bb}(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the completion of $\mathfrak{Bb}(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A})$.

Then any class $\mathbf{M}$ say, between $\mathbf{L}$ and $\mathbf{K}$ that is closed under forming subalgebras will not be atom–canonical, for $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}(\subseteq \mathbf{M})$, but $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A}) \notin \mathbf{K}(\supseteq \mathbf{M})$ because $\mathfrak{A} \notin \mathbf{M}$ and $\mathbf{SM} = \mathbf{M}$. We say, in this case, that $\mathbf{L}$ *is not atom–canonical with respect to* $\mathbf{K}$. This method is applied to $\mathbf{K} = \mathbf{S}\mathsf{Ra}\mathbf{CA}_l$, $l \geq 5$ and $\mathbf{L} = \mathsf{RRA}$ and to $\mathbf{K} = \mathsf{RRA}$ and $\mathbf{L} = \mathsf{RRA} \cap \mathsf{Ra}\mathbf{CA}_k$ for all $k \geq 3$ in, and will applied below to $\mathbf{K} = \mathbf{S}\mathsf{Nr}_n\mathbf{CA}_{n+k}$, $k \geq 3$ and $\mathbf{L} = \mathbf{RCA}_n$, where Ra denote the operator of forming relation algebra reducts (applied to classes) of $\mathbf{CA}$s, respectively.

Let $2 < n < m \leq \omega$. The notion of an algebra $\mathfrak{A}$ having signature $\mathbf{CA}_n$ possesing an m-square representation is defined for relation algebras by Hirsch and Hodkinson and can be easily extended to $\mathbf{CA}_n$s. An m-square representation only locally classic. Given $2 < l <$

$m \leq \omega$, an m-square representation is l-square but the converse may fail dramatically. An ω-square rpresentation–the limiting case-is an ordinary representation, such a representation is m-square for each finite m. Roughly, if we zoom in by a movable window to an m-square represention, there will come a point determined by the parameter m, were we mistake this locally classic represenation for a genuine ordinary Tarskian one. However, when we zoom out 'contradictions' reappear.

Theorem .24. *Let* $2 < n < \omega$ *and* $t(n) = n(n+1)/2+1$. *The variety* RCA_n *is not-atom canonical with respect to* $\mathbf{S}\mathsf{Nr}_n\mathbf{CA}_{t(n)}$. *In fact, there is a countable atomic simple* $\mathfrak{A} \in \mathsf{RCA}_n$ *such that* $\mathfrak{CmAt}\mathfrak{A}$ *does not have an* $t(n)$*-square,a fortiori* $t(n)$*- flat, representation.*

Consider the following statement: *There exists a countable, complete and atomic*

L_n *first order theory* T *in a signature* L, *meaning that the Tarski Lindenbuam quotient algebra* $\mathfrak{Fm}_T$ *is atomic, such that the type* Γ *consisting of co-atoms* $\mathfrak{Fm}_T$ *is realizable in every* m*–square model, but* Γ *cannot be isolated using* $\leq l$ *variables, where* $n \leq l < m \leq \omega$. A co-atom of $\mathfrak{Fm}_T$ is the negation of an atom in $\mathfrak{Fm}_T$. An m-square model of T is an m-square representation of $\mathfrak{Fm}_T$.

The last statement denoted by $\Psi(l, m)$, is short for Vaught's Theorem (VT*) fails at (the parameters)* l *and* m. Let VT(l, m) stand for VT *holds at* l *and* m, so that by definition $\Psi(l, m) \iff \neg\text{VT}(l, m)$. We also include $l = \omega$ in the equation by defining VT(ω, ω) as VT holds for $L_{\omega,\omega}$: Atomic countable first order theories have atomic countable models. It is well known that VT(ω,ω) is a direct consequence of the Orey-Henkin OTT. Recall that VT(ω,ω) is just Vaught's theorem, namely, countable atomic theories have atomic countable models.

Let $2 < n \leq l < m \leq \omega$. In VT$(l,m)$, while the parameter l measures how close we are to $L_{\omega,\omega}$, m measures the *'degree' of squareness* of permitted models. Using elementary calculas terminology one can view $\sum_{l\to\infty}$VT(l,ω) = VT(ω,ω) algebraically using ultraproducts as follows. Fix $2 < n < \omega$. For each $2 < n \leq l < \omega$, let R_l be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$, as defined on p.83 in §5 in the proof of Theorem 5.1 in "*Omitting types for finite variable fragments and complete representations of algebras.* **H. Andréka, I. Németi, and T. Sayed Ahmed–Journal of Symbolic Logic 73(1) (2008) p.65-89**" with l–blur (J_l, E_l) as defined in Definition 3.1 in op.cit and $f(l) \geq l$ as specified in Lemma 5.1 in op.cit (denoted by k therein). Let $\mathcal{R}_l = \mathfrak{Bb}(\mathsf{R}_l, J_l, E_l) \in \mathsf{RRA}$ where $\mathcal{R}_l$ is the relation algebra having atom structure denoted At in p. 73 in op.cit when the blown up and blurred algebra denoted R_l happens to be the finite Maddux algebra $\mathfrak{E}_{f(l)}(2,3)$ and let $\mathfrak{A}_l = \mathfrak{Nr}_n\mathfrak{Bb}_l(\mathsf{R}_l, J_l, E_l) \in \mathbf{RCA}_n$ as defined in Top of p.80 in op.cit(with $\mathsf{R}_l = \mathfrak{E}_{f(l)}(2,3)$). Then $(\mathsf{At}\mathcal{R}_l : l \in \omega \sim n)$, and $(\mathsf{At}\mathfrak{A}_l : l \in \omega \sim n)$ are sequences of weakly representable atom structures that are not strongly representable with a completely representable ultraproduct.

Let LCA_n denote the class of $\mathbf{CA}_n$s satisfying the Lyndon conditions which is the elementary closure of the class of completely representable $\mathbf{CA}_n$s.

We immediately get:

Corollary .25. *(Monk, Maddux, Biro, Hirsch and Hodkinson) Let $2 < n < \omega$. Then the set of equations using only one variable that holds in each of the varieties* RCA_n *and* RRA, *together with any finite first order definable expansion of each, cannot be derived from any finite set of equations valid in the variety. Furthermore,* LCA_n *is not finitely axiomatizable.*

Positive OTTs for L_n with standard 'unguarded' semantics

Unless otherwise specified, n will denote a finite ordinal > 2. Now we turn to proving omitting types theorems for certain (not all) L_n theories. But first a definition:

Definition .26. Let $\mathfrak{A} \in \mathbf{RCA}_n$ and let λ be a cardinal.

1. If $\mathbf{X} = (X_i : i < \lambda)$ is family of subsets of $\mathfrak{A}$, we say that $\mathbf{X}$ *is omitted in* $\mathfrak{C} \in \mathsf{Crs}_n$, if there exists an isomorphism $f : \mathfrak{A} \to \mathfrak{C}$ such that $\bigcap f(X_i) = \emptyset$ for all $i < \lambda$. When we want to stress the role of f, we say that $\mathbf{X}$ is omitted in $\mathfrak{C}$ via f.

2. If $X \subseteq \mathfrak{A}$ and $\prod X = 0$, then we refer to X as a *non-principal type* of $\mathfrak{A}$.

Observe that $\mathfrak{A} \in \mathbf{RCA}_n$ is completely representable $\iff$ $\mathfrak{A}$ is atomic, and the single non-principal type of co-atoms can be omitted in a Gs_n.

In the Theorem $n < \omega$:

Theorem .27. *Let $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathbf{CA}_\omega$ be countable. Let $\lambda < 2^\omega$ and let $\mathbf{X} = (X_i : i < \lambda)$ be a family of non-principal types of $\mathfrak{A}$. Then the following hold:*

1. *If* $\mathfrak{A} \in \mathsf{Nr}_n\mathbf{CA}_\omega$ *and the* X_i*s are non–principal ultrafilters, then* $\mathbf{X}$ *can be omitted in a* Gs_n*.*

2. *Every subfamily of* $\mathbf{X}$ *of cardinality* $< \mathfrak{p}$ *can be omitted in a* Gs_n*; in particular, every countable subfamily of* $\mathbf{X}$ *can be omitted in a* Gs_n*,*

3. *If* $\mathfrak{A}$ *is simple, then every subfamily of* $\mathbf{X}$ *of cardinality* $< \mathsf{covK}$ *can be omitted in a* Cs_n*,*

4. *It is consistent, but not provable (in* ZFC*), that* $\mathbf{X}$ *can be omitted in a* Gs_n*,*

5. *If* $\mathfrak{A} \in \mathsf{Nr}_n\mathbf{CA}_\omega$ *and* $|\mathbf{X}| < \mathfrak{p}$*, then* $\mathbf{X}$ *can be omitted* $\iff$ *every countable subfamily of* $\mathbf{X}$ *can be omitted.*

If $\mathfrak{A}$ is simple, we can replace $\mathfrak{p}$ by covK.

Definition .28. Let $\mathfrak{A} \in \mathbf{CA}_\beta$ and $\alpha < \beta$, then the α *neat reduct of* $\mathfrak{A}$ is the algebra obtained from $\mathfrak{A}$ by discarding operations in $\beta \sim \alpha$ and restricting the remaining operations to the set consisting only of α dimensional elements. An element is α dimensional if its dimension set, $\Delta x = \{i \in \beta : c_i x \neq x\}$ is contained in α. Such an algebra is denoted by $\mathsf{Nr}_\alpha\mathfrak{A}$.

We show (algebraically) that the maximality condition cannot be removed when we consider uncountable theories.

Theorem .29. *Let κ be an infinite cardinal. Then there exists an atomless $\mathfrak{C} \in \mathbf{CA}_\omega$ such that for all $2 < n < \omega$, $|\mathfrak{Nr}_n\mathfrak{C}| = 2^\kappa$, $\mathfrak{Nr}_n\mathfrak{C} \in \mathsf{LCA}_n (= \mathbf{El}_n)$, but $\mathfrak{Nr}_n\mathfrak{C}$ is not completely representable.*

Thus the non–principal type of co–atoms of $\mathfrak{Nr}_n\mathfrak{C}$ cannot be omitted. In particular, the condition of maximality cannot be removed.

Since $\mathsf{Nr}_n\mathbf{CA}_\omega \subseteq \mathsf{LCA}_n = \mathbf{El}\mathsf{CRCA}_n$

Corollary .30. *(Hirsch Hodkinson) For $2 < n < \omega$, the classes CRCA_n and CRRA are not elementary.*

We have proved through OTT **theorems in a somewhat short path– that the class of representable algebras of finite dimension > 2 is not finitely axiomatizable, while the class of completely representable algebras of the same dimension is not first order definable–two cornerstones of the theory of cylindric algebras that took dozens (if not perhaps hundreds of publications) to prove that**

can be traced back to Monk's paper on non finite axiomatizability of relation algebras using Lyndon algebras in the mid sixties of the last century, to his 1969 JSl paper on CAs, refined further by Maddux, Andréka, Hirsch, Hodkinson, Sági, Sayed Ahmed and others..

.Also positive results obtained by circumventing such negative results are obtained by Sain, Németi and Sayed Amed for first order with and without equality using finitely presented semigroups, an idea that can be traced to Craig.

I will stop here..unless..

More non-elementary classes

Definition .31. Fix finite $n > 2$ and assume that $\mathfrak{A} \in \mathbf{CA}_n$ is atomic.

(1) An n–*dimensional atomic network* on $\mathfrak{A}$ is a map $N : {}^n\Delta \to At\mathfrak{A}$, where Δ is a non–empty set of *nodes*, denoted by (N), satisfying the following consistency conditions for all $i < j < n$:

- If $\bar{x} \in {}^n(N)$ then $N(\bar{x}) \leq \mathsf{d}_{ij} \iff x_i = x_j$,
- If $\bar{x}, \bar{y} \in {}^n(N)$, $i < n$ and $\bar{x} \equiv_i \bar{y}$, then $N(\bar{x}) \leq \mathsf{c}_i N(\bar{y})$.

For n–dimensional atomic networks M and N, we write $M \equiv_i N \iff M(\bar{y}) = N(\bar{y})$ for all $\bar{y} \in {}^n(n \sim \{i\})$.

(2) Assume that $m, k \leq \omega$. The *atomic game* $G_k^m(\mathsf{At}\mathfrak{A})$, *or simply* G_k^m, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds where is offered only one move, namely, *a cylindrifier move*: Suppose that we are at round $t > 0$. Then picks a previously played network N_t ($(N_t) \subseteq m$), $i < n$, $a \in \mathsf{At}\mathfrak{A}$, $x \in {}^n(N_t)$, such that $N_t(\bar{x}) \leq \mathsf{c}_i a$. For her response, has to deliver a network M such that $(M) \subseteq m$, $M \equiv_i N$, and there is $\bar{y} \in {}^n(M)$ that satisfies $\bar{y} \equiv_i \bar{x}$ and $M(\bar{y}) = a$. We write $G_k(\mathsf{At}\mathfrak{A})$, or simply G_k, for $G_k^m(\mathsf{At}\mathfrak{A})$ if $m \geq \omega$.

(3) The ω–rounded game $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G_\omega^m(\mathsf{At}\mathfrak{A})$ except that has the option to reuse the m nodes in play.

Lemma .32. *Let* $2 < n < m \leq \omega$.

1. *If* $\mathfrak{A} \in \mathbf{CA}_n$ *is finite and* $\mathfrak{A}$ *has an* m*-square representation, then has a winning strategy in* $G^m(\mathsf{At}\mathfrak{A})$

2. *If* $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathbf{CA}_m$*, then has a winning strategy in* $\mathbf{G}^m(\mathsf{At}\mathfrak{A})$*.*

In our proof we use a variation on a *rainbow constructions*. Fix $2 < n < \omega$. Given relational structures G (the greens) and R (the reds) the rainbow atom structure of a $\mathbf{CA}_n$ consists of equivalence classes of surjective maps $a : n \to \Delta$, where Δ is a coloured graph. A *coloured graph* is a complete graph labelled by the rainbow colours, the greens $\mathsf{g} \in \mathsf{G}$, reds $\mathsf{r} \in \mathsf{R}$, and whites; and some $n-1$ tuples are labelled by 'shades of yellow'. In coloured graphs certain triangles are not allowed for example all green triangles are forbidden. A red triple $(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*})$ $i, j, j', k', i^*, k^* \in \mathsf{R}$ is not allowed, unless $i = i^*$, $j = j'$ and $k' = k^*$, in which case we say that the red indices match. The equivalence relation relates two such maps $\iff$ they essentially define the same graph. We let $[a]$ denote the equivalence class containing a. For $2 < n < \omega$, we use the graph version of the usual atomic ω–rounded game $G^m_\omega(\alpha)$ with m nodes, played on atomic networks of the $\mathbf{CA}_n$ atom structure α. The game $\mathbf{G}^m(\beta)$ where β is a $\mathbf{CA}_n$ atom structure is like $G^m_\omega(\mathsf{At}\mathfrak{A})$ except that has the option to reuse the m nodes in play. We use the 'graph versions' of these games. The typical winning strategy for is bombading with cones having green tints and a common base until she runs out of consistent triples of reds. The (complex) rainbow algebra based on G and R is denoted by $\mathfrak{A}_{\mathsf{G},\mathsf{R}}$. The dimension n will always be clear from context.

Lemma .33. *Let α be a countable atom structure. There is a k rounded atomic game with $k \leq \omega$ (played on atomic networks) such that if has a winning strategy in $\mathbf{H}_\omega(\alpha)$, then any algebra having atom structure α is completely representable and there exists a complete $\mathfrak{D} \in \mathbf{RCA}_\omega$ such that $\mathfrak{Cm}\alpha \cong \mathsf{Nr}_n\mathfrak{D}$ and $\alpha \cong \mathsf{AtNr}_n\mathfrak{D}$. In particular, $\mathfrak{Cm}\alpha \in \mathsf{Nr}_n\mathbf{CA}_\omega$ and $\alpha \in \mathsf{AtNr}_n\mathbf{CA}_\omega$.*

Lemma .34. *Any class K between $\mathbf{S}_d\mathsf{Nr}_n\mathbf{CA}_\omega \cap$ CRCA_n and $\mathbf{S}_c\mathsf{Nr}_n\mathbf{CA}_{n+3}$ is not elementary*

Proof. **(1) has a winning strategy in $\mathbf{G}_{n+3}(\mathsf{At}\mathfrak{C})$ for a rainbow-like algebra $\mathfrak{C}$:**

Take the a rainbow–like $\mathbf{CA}_n$, call it $\mathfrak{C}$, based on the ordered structure $\mathbb{Z}$ and $\mathbb{N}$. The reds R is the set $\{\mathsf{r}_{ij} : i < j < \omega(=$

$\mathbb{N})\}$ and the green colours used constitute the set $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like the usual rainbow constructions based on $\mathbb{Z}$ and $\mathbb{N}$, but now the triple $(\mathsf{g}_0^i, \mathsf{g}_0^j, \mathsf{r}_{kl})$ is also forbidden if $\{(i,k),(j,l)\}$ is not an order preserving partial function from $\mathbb{Z} \to \mathbb{N}$. It can be shown that has a winning strategy in the graph version of the game $\mathbf{G}^{n+3}(\mathsf{At}\mathfrak{C})$ played on coloured graphs. The rough idea here, is that, as is the case with winning strategy's of in rainbow constructions, bombards with cones having distinct green tints demanding a red label from to appexes of succesive cones. The number of nodes are limited but has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces to choose red labels, one of whose indices form a decreasing sequence in $\mathbb{N}$. In ω many rounds forces a win, so $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathbf{CA}_{n+3}$.

(2) has a winning strategy in $\mathbf{H}_k(\mathsf{At}\mathfrak{C})$ for all $k < \omega$:

It can be shown that for $k < \omega$, has a winning strategy in $G_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ inspite of the newly forbidden triple consisting of two greens and one red, synchronized by an order preserving function. This plainly makes her choices more restricted. But we can go further. It can be shown with some more effort (but not much more) that, in fact, has a winning strategy in even the stronger game $\mathbf{H}_k(\mathsf{At}\mathfrak{C}_{\mathbb{Z},\mathbb{N}})$ for all $k < \omega$.

(3) Finishing the proof: All games used are deterministic. For each $k < \omega$, let σ_k describe the winning strategy of

$\mathbf{H}_k(\alpha)$. Let $\mathfrak{C} = \mathfrak{Tm}\alpha$. Let $\mathfrak{D}$ be a non–principal ultrapower of $\mathfrak{C}$. Then has a winning strategy σ in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$ — essentially she uses σ_k in the k'th component of the ultraproduct so that at each round of $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$, is still winning in co–finitely many components, this suffices to show she has still not lost. Now one can use an elementary chain argument to construct countable elementary subalgebras $\mathfrak{C} = \mathfrak{A}_0 \preceq \mathfrak{A}_1 \preceq \ldots \preceq \ldots \mathfrak{D}$ in the following way. One defines $\mathfrak{A}_{i+1}$ to be a countable elementary subalgebra of $\mathfrak{D}$ containing $\mathfrak{A}_i$ and all elements of $\mathfrak{D}$ that σ selects in a play of $\mathbf{H}_\omega(\mathsf{At}\mathfrak{D})$ in which only chooses elements from $\mathfrak{A}_i$. Now let $\mathfrak{B} = \bigcup_{i<\omega} \mathfrak{A}_i$. This is a countable elementary subalgebra of $\mathfrak{D}$, hence necessarily atomic, and has a winning strategy in $\mathbf{H}_\omega(\mathsf{At}\mathfrak{B})$. . So by te previous Lemma (using $\mathsf{At}\mathfrak{B}$ in place of α), we get that $\mathfrak{CmAt}\mathfrak{B} \in \mathsf{Nr}_n\mathbf{CA}_\omega$.

Since $\mathfrak{B} \subseteq_d \mathfrak{Cm}\mathsf{At}\mathfrak{B}$, then $\mathfrak{B} \in \mathbf{S}_d\mathsf{Nr}_n\mathbf{CA}_\omega$ and by Lemma .33, we also have that $\mathfrak{B} \in \mathsf{CRCA}_n$. But has a wining strategy in $\mathbf{G}^m(\mathsf{At}\mathfrak{B})$, $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathbf{CA}_m$. To finalize the proof, let $\mathbf{K}$ be as given. Then $\mathfrak{B} \equiv \mathfrak{C}$, $\mathfrak{B} \in \mathbf{K}(\subseteq \mathbf{S}_d\mathsf{Nr}_n\mathbf{CA}_\omega \cap \mathsf{CRCA_n})$, but $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathbf{CA}_{n+3}(\supseteq \mathbf{K})$ giving that $\mathbf{K}$ is not elementary.

$\square$

Theorem .35. *1. (Hirsch and Hodkinson using Erdos probabablistic graphs): There is a finite $k \geq 2$, such that for all $m \geq n+k$ the class of frames $\mathsf{Str}(\mathbf{S}\mathsf{Nr}_n\mathbf{CA}_m) = \{: \mathfrak{Cm} \in \mathbf{S}\mathsf{Nr}_n\mathbf{CA}_m\}$ is not elementary. An entirely analogous result holds for $\mathbf{RA}$s,*

2. Let $\mathbf{O} \in \{\mathbf{S}_c, \mathbf{S}_d, \mathbf{I}\}$ and $k \geq 3$. Then the class of frames $\mathsf{K}_k = \{: \mathfrak{Cm} \in \mathbf{O}\mathsf{Nr}_n\mathbf{CA}_{n+k}\}$ is not elementary.