Spiralling Domains in Dimension 2

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(Joint work in progress with Xavier Buff)

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Goal: understand the parabolic basin $\mathscr{B}_0$ of $F\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $F(0) = 0$ and $D_0 F = \mathrm{id}$

$$\mathscr{B}_0 = \{z \in \mathbb{C}^n \mid F^{\circ m}(z) \xrightarrow[m\to+\infty]{} 0\}$$

Parabolic domain: connected component P of $\mathring{\mathscr{B}}_0$, $F(P) \subset P$.

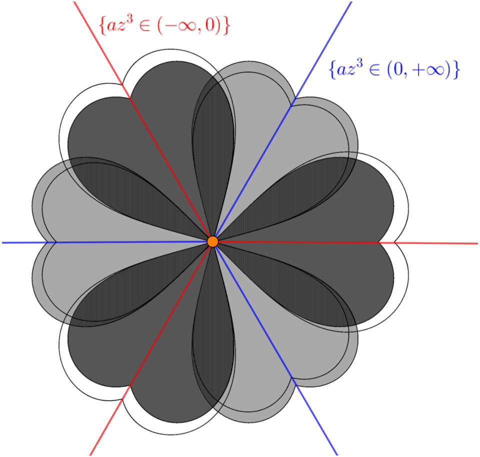
Theorem (Buff-R., in progress)

There exist $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ polynomial maps tangent to the identity at the origin with infinitely many parabolic domains of spiralling type.

Dimension 1

- $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ tangent to the identity and $f \neq \mathrm{id}$:

$$f(z) = z + az^{k+1} + \mathrm{O}(z^{k+2}) \quad \text{with} \quad a \in \mathbb{C} \setminus \{0\}.$$



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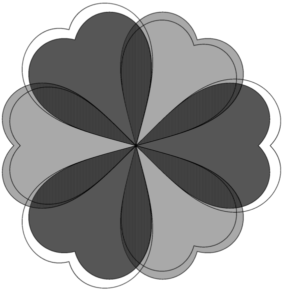

$k = 3$

f is topologically conjugate to the time-1 flow of $z^{k+1} \frac{\partial}{\partial z}$.

Dimension 2

Setting:

- $\mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$
- $F(\mathbf{z}) = \mathbf{z} + v(\mathbf{z}) + \mathrm{O}\left(\|\mathbf{z}\|^{k+2}\right)$, $k \geq 1$
- $v : \mathbb{C}^2 \to \mathbb{C}^2$ homogeneous map of degree $k + 1$.

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Idea: Look at $\vec{v}(\mathbf{z})$

- search for preferred directions for the dynamics
- understand orbits of f using real time trajectories of $\vec{v}(\mathbf{z})$.

$$F(\mathbf{z}) = \mathbf{z} + v(\mathbf{z}) + \mathrm{O}\left(\|\mathbf{z}\|^{k+2}\right), \quad F^{\circ n}(\mathbf{z}) = \mathbf{z}_n$$

Fact: $\mathbf{z}_n \to \mathbf{0}$ tangentially to $[\mathbf{t}] \in \mathbb{P}^1(\mathbb{C}) \Longrightarrow \exists \lambda \in \mathbb{C}$ s.t. $v(\mathbf{t}) = \lambda \mathbf{t}$.

- $[\mathbf{t}] \in \mathbb{P}^1(\mathbb{C})$ is a characteristic direction if $v(\mathbf{t}) = \lambda \mathbf{t}$. $[\mathbf{t}]$ is non-degenerate if $\lambda \neq 0$, degenerate if $\lambda = 0$.
- v is dicritical if all directions are characteristic, non-dicritical otherwise.

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From now on: v non-dicritical

Maps tangent to the identity in dimension 2

Assumptions:

- $\vec{v}$ is a homogeneous vector field of degree $k+1$ on $\mathbb{C}^2$:

$$\vec{v} := U\partial_x + V\partial_y$$

  with U and V homogeneous polynomials of degree $k+1$;
-
$$\Phi := xV - yU$$

  vanishes on $k+2$ *characteristic directions*, counting multiplicities;
-
$$F(\boldsymbol{x}) = \boldsymbol{x} + \vec{v}(\boldsymbol{x}) + \mathrm{O}\left(\|\boldsymbol{x}\|^{k+2}\right).$$

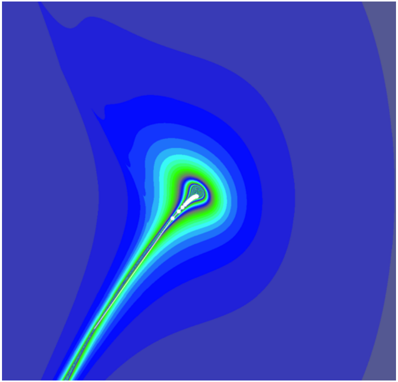
Observation:

- Near $\mathbf{0}$, orbits of F shadow real-time trajectories of $\vec{v}$.

Known results

Theorem (Écalle, Hakim, Abate, . . . , López-Hernanz, Rosas)

For any F, tangentially to each characteristic direction, there is either a curve of fixed points, or at least one parabolic petal.

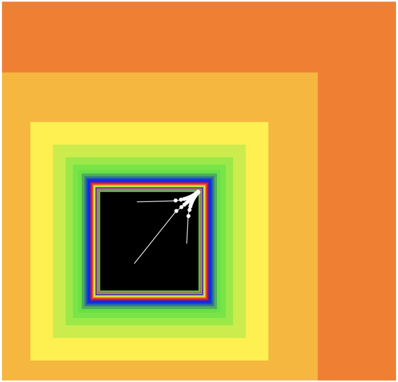


$$F(x, y) = (x + y^2 + x^3, y + x^2)$$

Known results

Proposition (Écalle, Hakim)

Existence of F which have *parabolic domains* on which orbits converge to **0** tangentially to a characteristic direction.



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Existence of F which have *parabolic domains* on which orbits converge to $\mathbf{0}$ *spiralling* around a characteristic direction.

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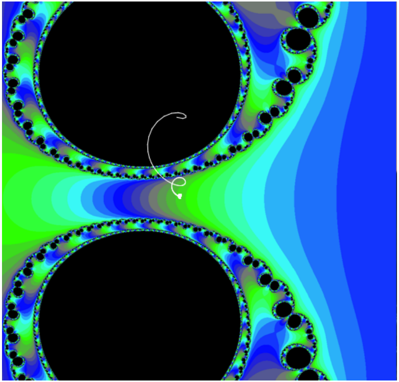

$$F(x, y) = (x - x^2, y + y^2 + 4x^2)$$

Example by Astorg and Boc-Thaler

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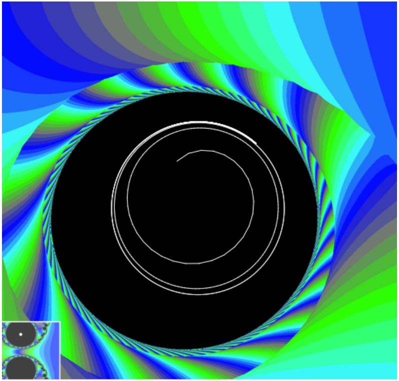

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Spiralling domains in dimension 2

Theorem (Buff-R., in progress)

For $a \in \mathbb{R} \setminus \{\mathbf{0}\}$, the polynomial endomorphism $F_a : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$F_a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + a \begin{pmatrix} x(x-y) \\ y(x-y) \end{pmatrix}$$

has infinitely many spiralling domains contained in distinct invariant Fatou components.

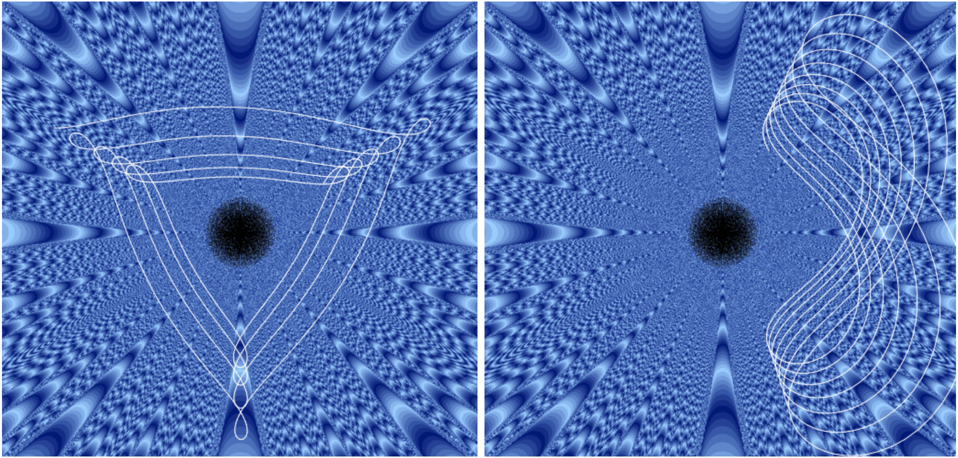
Tools

- homogeneous vector fields;
- affine surfaces;
- triangular billiards.

The family F_a

$$F_a\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} y^2 \\ x^2 \end{pmatrix} + a\begin{pmatrix} x(x-y) \\ y(x-y) \end{pmatrix}$$

The dynamics of F_a for $a = 0$



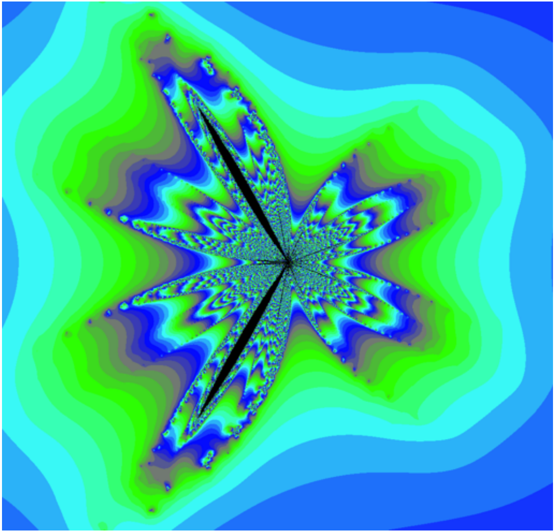
Trajectories for $\vec{v} = y^2 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y}$

- $\vec{v}$ is a Hamiltonian vector field
- Complex trajectories of $\vec{v}$:

$$\mathscr{S}_\kappa := \left\{ \mathbf{z} \in \mathbb{C}^2 \mid \Phi(\mathbf{z}) := x^3 - y^3 = \kappa \right\} \text{ with } \kappa \in \mathbb{C}.$$

- $\mathscr{S}_0 = \{y = x\} \cup \{y = jx\} \cup \{y = j^2 x\}$ with $j = e^{\frac{2\pi i}{3}}$
- $\mathbf{0} \notin \overline{\mathscr{S}_\kappa}$ for $\kappa \neq 0$, and so real trajectories of $\vec{v}$ in $\mathscr{S}_\kappa$ do not converge to $\mathbf{0}$.
- For $\kappa \neq 0$, $\mathscr{S}_\kappa \simeq \mathrm{Torus} \setminus \{3 \text{ points}\}$, on which $\vec{v}$ is a translation vector field.
- If $\kappa = (p + jq)^3 r$, with $(p, q) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ and $r \in \mathbb{R} \setminus \{0\}$, then the real trajectories of $\vec{v}$ are periodic, that is closed.

The dynamics of F_a for $a = 0.1$



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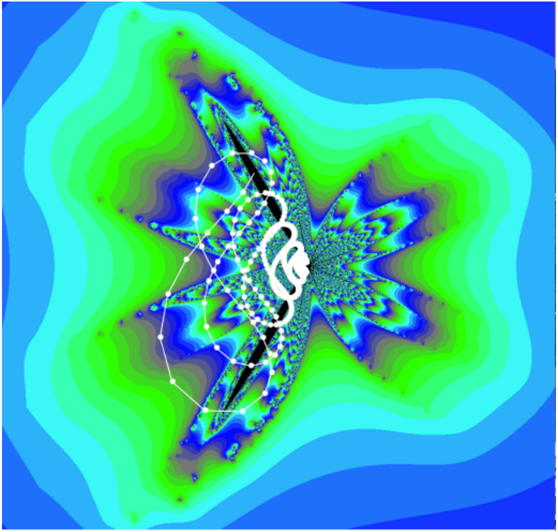

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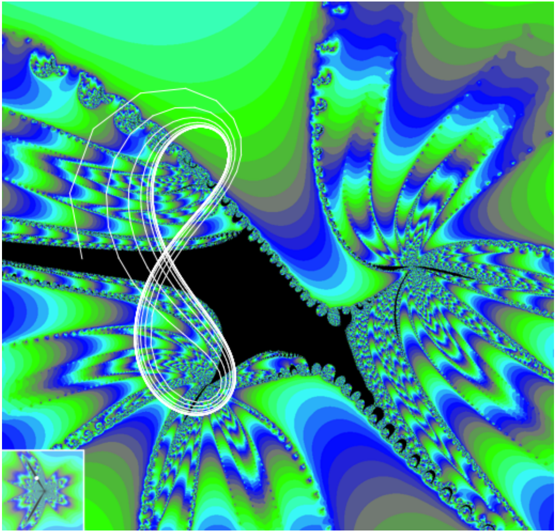

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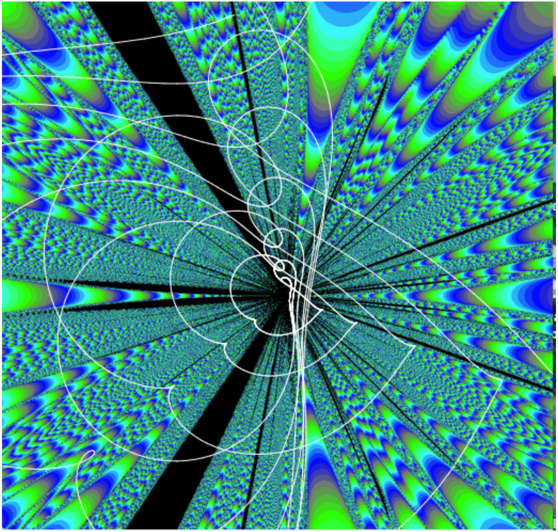

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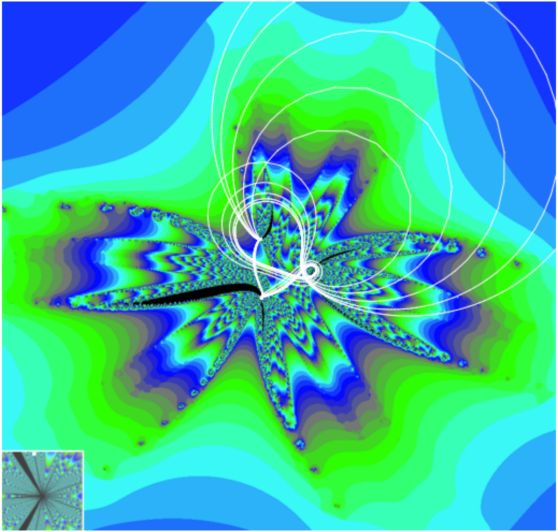

Dynamics of homogeneous vector fields

- A trajectory for $\vec{\boldsymbol{v}}$ is a solution of the differential equation

$$\dot{\gamma} = \vec{\boldsymbol{v}} \circ \gamma.$$

- Complex-time trajectories are Riemann surfaces which cover $\mathbb{CP}^1$ minus the characteristic directions.
- What does the projection to $\mathbb{CP}^1$ of a real-time trajectory look-like?

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- What does the projection to $\mathbb{CP}^1$ of a real-time trajectory look-like?

Proposition (Abate-Tovena)

We may equip $\mathbb{CP}^1$ with the structure of an affine surface $\mathbf{S}_{\vec{v}}$ so that the projection to $\mathbf{S}_{\vec{v}}$ of real-time trajectories of $\vec{\boldsymbol{v}}$ are geodesics.

Affine surfaces and geodesics

Definition (Affine surface)

An *affine surface* **S** is a Riemann surface whose change of charts are affine maps $z \mapsto \lambda z + \mu$ with $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$.

Example: **C** is the complex plane with its canonical affine structure.

Definition (Affine map)

A map between affine surfaces is an *affine map* if its expression in affine charts is of the form $z \mapsto \lambda z + \mu$.

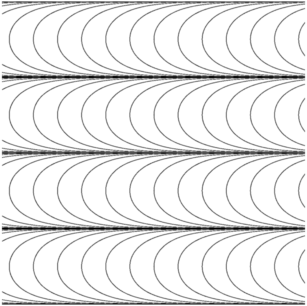
Definition (Geodesic)

A curve $\delta : I \to$ **S** defined on an interval $I \subseteq \mathbb{R}$ is a *geodesic* if δ is the restriction of an affine map $\varphi : U \to$ **S** defined on an open subset $U \subseteq$ **C**.

An example

- The dilation plane $\widetilde{\mathbf{C}}$ with underlying Riemann surface $\mathbb{C}$, whose affine charts are the restrictions of

$$\exp(z) : \widetilde{\mathbf{C}} \to \mathbf{C} \setminus \{0\}.$$



A family of parallel geodesics in $\widetilde{\mathbf{C}}$.

Nonlinearity

- The nonlinearity of a holomorphic map $\varphi : \mathbf{S} \to \mathbf{T}$ with non vanishing derivative is the 1-form $\mathcal{N}_\varphi$ defined on **S** by

$$\mathcal{N}_\varphi := \mathrm{d}(\log \varphi') = \frac{\mathrm{d}\varphi'}{\varphi'}.$$

- $\mathcal{N}_\varphi = 0$ if and only if φ is an affine map.
- If $\varphi : \mathbf{S} \to \mathbf{T}$ and $\psi : \mathbf{T} \to \mathbf{U}$ are holomorphic maps, then

$$\mathcal{N}_{\psi\circ\varphi} = \mathcal{N}_\varphi + \varphi^*(\mathcal{N}_\psi).$$

Affine surface of a homogeneous vector field

- $\vec{v} = U\partial_x + V\partial_y$ is homogeneous of degree $k+1$.
- $z : \mathbb{CP}^1 \ni [x : y] \mapsto \dfrac{x}{y} \in \widehat{\mathbb{C}}$.
- $f\left(\dfrac{x}{y}\right) = \dfrac{U(x,y)}{V(x,y)}$.
- $p\left(\dfrac{x}{y}\right) = \dfrac{xU(x,y) - yV(x,y)}{y^{k+2}}$.

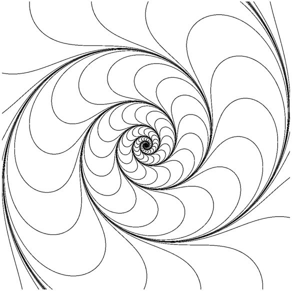
Proposition

The nonlinearity of $z : \mathbf{S}_{\vec{v}} \to \mathbf{C}$ is

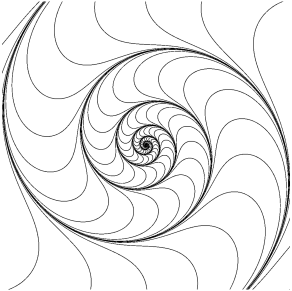
$$\nu := \left(\frac{p'(z)}{p(z)} - \frac{k}{z - f(z)}\right) \mathrm{d}z.$$

Affine surface of a homogeneous vector field

- Singularities of ν are characteristic directions.
- Assume there is a simple pole and let ρ be the residue.



$\mathrm{Re}(\rho) > 1$



$\mathrm{Re}(\rho) < 1$

Theorem (Écalle, Hakim)

If ν has a simple pole and $\mathrm{Re}(\rho) > 1$, there is a parabolic domain on which orbits converge to **0** *tangentially to the characteristic direction.*

Affine surface of a homogeneous vector field

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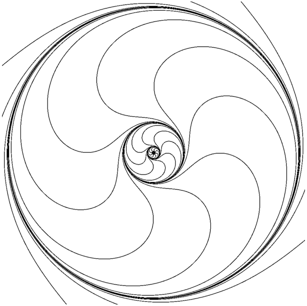
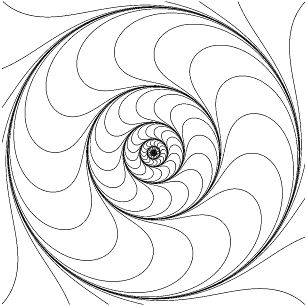


$\rho = 1 - 2\mathrm{i}$



$\rho = 1 - 4\mathrm{i}$

Proposition (Rivi,Rong)

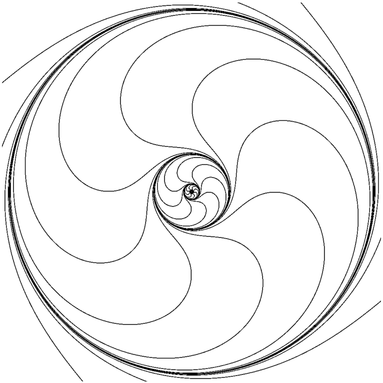
If ν has a simple pole and $\mathrm{Re}(\rho) = 1$, there is a parabolic domain on which orbits converge to **0** spiralling around the characteristic direction.

Closed geodesics

- A geodesic $\delta : I \to \mathbf{S}$ is *closed* if there exists $\lambda \in (0, +\infty)$ and $t_0 < t_1$ in I such that

$$\delta(t_1) = \delta(t_0) \quad \text{and} \quad \dot{\delta}(t_1) = \lambda \dot{\delta}(t_0).$$

- Such a geodesic is *attracting* if $\lambda \in (0, 1)$.



Spiralling domains

- If an affine surface contains an attracting closed geodesic, it contains an *attracting dilation cylinder* foliated by attracting closed geodesic.

Proposition (Buff-R., in progress)

Assume $F(\boldsymbol{x}) = \boldsymbol{x} + \vec{\boldsymbol{v}}(\boldsymbol{x})$ with $\vec{\boldsymbol{v}}$ homogeneous. If $\mathbf{S}_{\vec{v}}$ contains an attracting dilation cylinder $\mathcal{C}$, then F has a spiralling domain in which orbits converge to $\mathbf{0}$, spiralling towards an attracting closed geodesic of $\mathcal{C}$.

Proposition (Buff-R.)

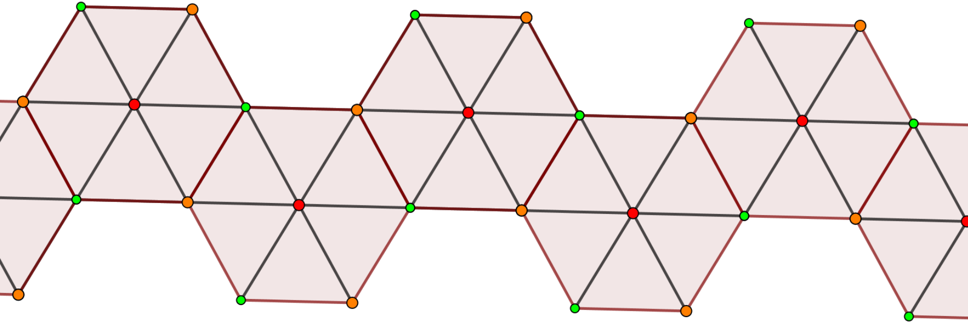
Assume $a \in \mathbb{R} \setminus \{0\}$ and

$$\vec{\boldsymbol{v}} := \big(y^2 + ax(x-y)\big)\partial_x + \big(x^2 + ay(x-y)\big)\partial_y.$$

Then, $\mathbf{S}_{\vec{v}}$ contains infinitely many non homotopic attracting dilation cylinders.

Polygonal models

- If
$$\vec{\boldsymbol{v}} = y^2\partial_x + x^2\partial_y,$$
the affine surface $\mathbf{S}_{\vec{v}}$ may be obtained by gluing equilateral triangles.



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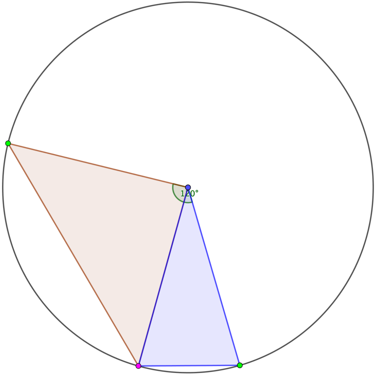

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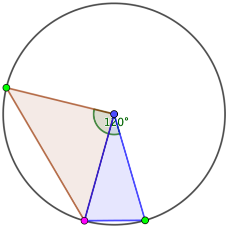

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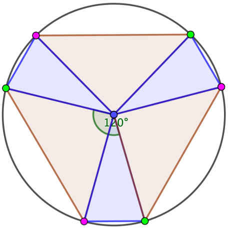

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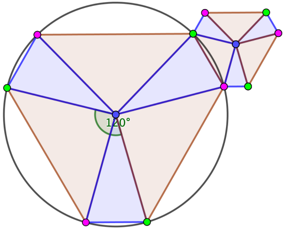

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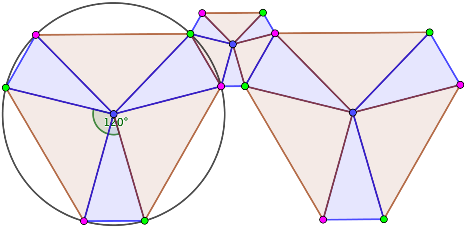

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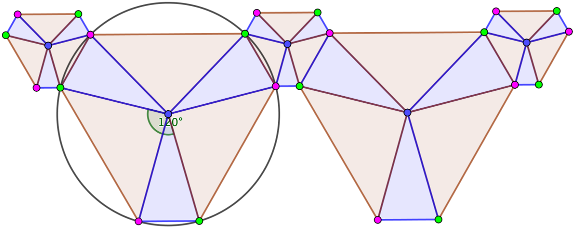

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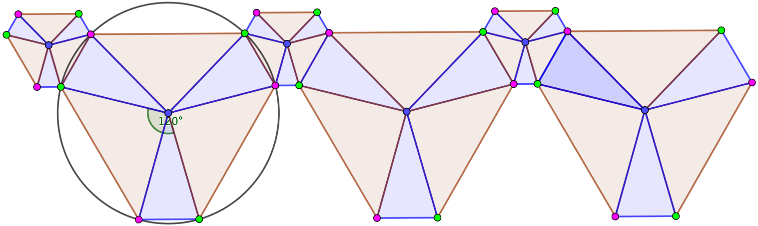
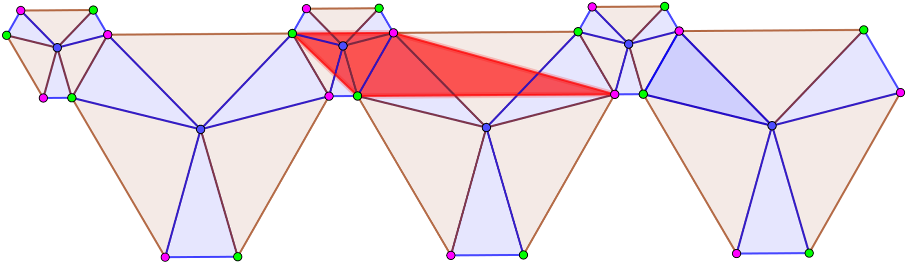
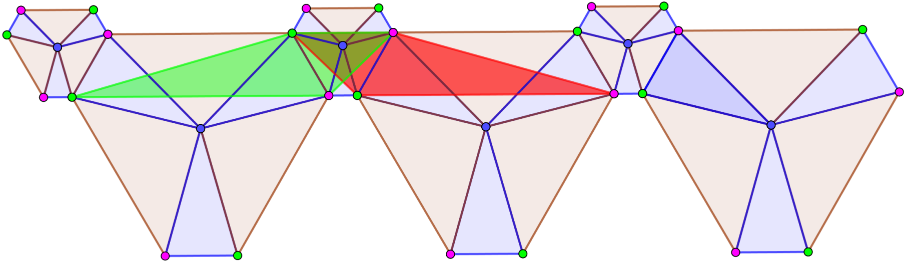


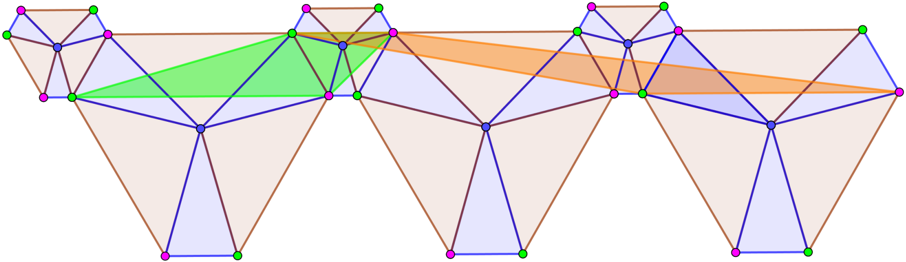
One attracting cylinder



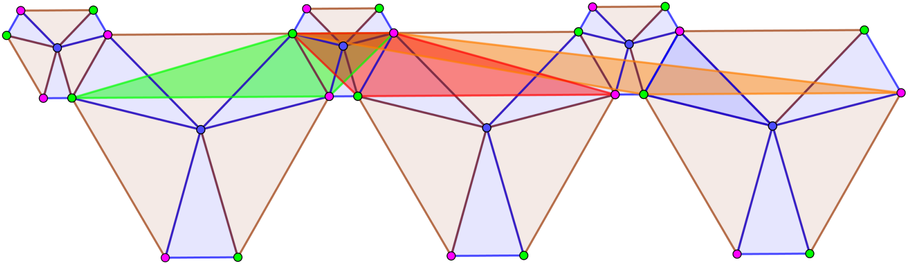
A second attracting cylinder



A third attracting cylinder



Three attracting cylinders



Happy Birthday László!