(Generalized) Descriptive Set Theory meets Model Theory

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Model Theoretic Logics and their Frontiers
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The number of models of a theory

Let T be a countable complete theory admitting infinite models, and κ be an infinite cardinal.

$$I(\kappa, T) = \text{number of non-isomorphic } \kappa\text{-sized models of } T.$$

In particular: "$I(\kappa, T) = 1$" $\rightsquigarrow$ **κ-categorical**

The spectrum problem

Describe the possible behaviors of $I(\kappa, T)$ as a function of κ.

Solved (for uncountable κ's) by the model-theorists Löwenheim-Skolem, Morley, Shelah, Hart-Hrushovski-Laskowski.

The number of models of a theory

Löwenheim-Skolem theorem

$I(\kappa, T) \neq 0$ *for some* infinite κ iff $I(\kappa, T) \neq 0$ *for all* infinite κ's.

Morley's categoricity theorem

$I(\kappa, T) = 1$ *for some* uncountable κ iff $I(\kappa, T) = 1$ *for all* uncountable κ's.

This divides our theories T in four groups:

- **totally categorical**: κ-categorical for all infinite κ
- **uncountably categorical**: κ-categorical for all uncountable κ
- **countably categorical**: $\aleph_0$-categorical
- **never categorical**: not κ-categorical for any infinite κ

The number of models of a theory

What are the other possible values for $I(\kappa, T)$ when T is not κ-categorical?

Obvious bounds: $1 < I(\kappa, T) \leq 2^\kappa$... can we say more?

Solution for uncountable κ's: more dividing lines (= **stability theory**)!

Shelah's Main Gap Theorem

Let $\kappa \geq \aleph_1$ be the γ-th cardinal.

1. If T is classifiable shallow of depth α, then $I(\kappa, T) \leq \beth_\alpha \left(|\gamma|^{2^{\aleph_0}} \right)$.
2. If T is not classifiable shallow, then $I(\kappa, T) = 2^\kappa$.

Remark: If T is classifiable shallow, its depth α is countable (Lascar).

Building on Shelah's work, Hart-Hrushovski-Laskowski gave a complete solution to the spectrum problem for countable complete first-order theories by listing all possible spectra.

The number of models of a theory

What if $\kappa = \aleph_0$?

Theorem (Vaught)

$I(\aleph_0, T) \neq 2$.

In contrast, for every $n \in \mathbb{N} \setminus \{2\}$ there are T such that $I(\aleph_0, T) = n$.

Theorem (Morley)

$I(\aleph_0, T) \leq \aleph_0$, or $I(\aleph_0, T) = \aleph_1$, or else $I(\aleph_0, T) = 2^{\aleph_0}$.

The values $\aleph_0$ and $2^{\aleph_0}$ can actually be attained.

Vaught conjecture (still open)

There is no T such that $I(\aleph_0, T) = \aleph_1$ (assuming $2^{\aleph_0} > \aleph_1$).

Entering (classical) descriptive set theory...

> "Descriptive set theory is the study of **definable sets** in **Polish spaces**".
>
> A. Kechris, *Classical descriptive set theory*, 1995

Polish spaces: Completely metrizable spaces of weight ω.

Examples: $\mathbb{R}^n$, $\mathbb{C}^n$, all separable Banach spaces. The Cantor space ${}^{\omega}2$ and the Baire space ${}^{\omega}\omega$, where ${}^{\omega}A$ is equipped with the topology generated by $\boldsymbol{N}_s = \{x \in {}^{\omega}A \mid s \sqsubseteq x\}$ for $s \in {}^{<\omega}A$.

Definable sets:
- *Borel* sets $\rightsquigarrow$ ω_1-algebra generated by the topology
- *analytic* sets $\rightsquigarrow$ continuous images of Polish spaces
- *coanalytic* sets $\rightsquigarrow$ complements of analytic sets,

and so on.

Part of the success experienced by classical DST is arguably due to the strong structural results obtained, and to its wide applicability in general mathematics, including model theory.

DST and model theory

Countable models of a given first-order theory T can be coded as elements of ${}^{\omega}2$ via characteristic functions. (For example, a graph on ω can be naturally identified with the characteristic function of its edge relation, which is an element of ${}^{\omega\times\omega}2 \approx {}^{\omega}2$.)

$$\begin{aligned} \mathrm{Mod}^{\omega}_{T} &\quad\rightsquigarrow\quad \text{(codes of) countable models of } T \\ \cong^{\omega}_{T} &\quad\rightsquigarrow\quad \text{isomorphism relation on } \mathrm{Mod}^{\omega}_{T} \end{aligned}$$

Key point: $\mathrm{Mod}^{\omega}_{T} \subseteq {}^{\omega}2$ is a *Borel* subset of ${}^{\omega}2$, and $\cong^{\omega}_{T}$ is an *analytic* equivalence relation (i.e. an analytic subset of $(\mathrm{Mod}^{\omega}_{T})^2$).

Question

Is the isomorphism relation $\cong^{\omega}_{T}$ Borel? If yes, what is its Borel rank?

Structural results: from DST to model theory...

Theorem

Let E be an equivalence relation a Polish space X.

Silver's Dichotomy: If E is coanalytic, then either E has $\leq \aleph_0$-many classes, or else there is a (closed) copy of ${}^{\omega}2$ inside X consisting of pairwise E-inequivalent elements.

Burgess' Trichotomy: If E is analytic, then either E has $\leq \aleph_0$-many classes, or it has $\aleph_1$-many classes, or else there is a (closed) copy of ${}^{\omega}2$ inside X consisting of pairwise E-inequivalent elements.

Corollary

- Either $I(\aleph_0, T) \leq \aleph_0$, or $I(\aleph_0, T) = \aleph_1$, or else $I(\aleph_0, T) = 2^{\aleph_0}$. (*Morley*)
- If $\cong^{\omega}_{T}$ is Borel, then either $I(\aleph_0, T) \leq \aleph_0$, or else $I(\aleph_0, T) = 2^{\aleph_0}$.

Remark: With the DST approach, the corollary remains true even if we consider more general theories, e.g. $\mathcal{L}_{\omega_1\omega}$-theories!

...and backwards, from model theory to DST

L. Mayer proved that if T is o-minimal, then either $I(\aleph_0, T) < \aleph_0$ or $I(\aleph_0, T) = 2^{\aleph_0}$.

Building on her work and exploiting a model-theoretic analysis of the types of T, Rast-Sahota obtained very interesting DST-complexity information on $\cong^\omega_T$ for such theories.

Theorem (Rast-Sahota)

Let T be o-minimal.

1. If T has no nonsimple types, then $\cong^\omega_T$ is Borel reducible to F_2 (i.e. the ctbl models of T can be classified in a Borel fashion using countable sets of reals as invariants).
2. If T admits a nonsimple type, then $\cong^\omega_T$ is S_∞-complete.

In the former case $\cong^\omega_T$ is Borel, while in the latter $\cong^\omega_T$ is analytic complete (hence not Borel).

Example: If T defines an infinite group, then $\cong^\omega_T$ is S_∞-complete and hence not Borel.

Slogan: "Replace ω with $\kappa > \omega$ or $\mathrm{cof}(\kappa)$ in all classical definitions and statements."

Given two cardinals ν, μ, the *bounded topology* τ_b on the space ${}^{\nu}\mu = \{f \mid f \colon \nu \to \mu\}$ is the one generated by the sets of the form $\boldsymbol{N}_s = \{x \in {}^{\nu}\mu \mid s \sqsubseteq x\}$ for $s \in {}^{<\nu}\mu$.

Generalized Cantor space	**Generalized Baire space**
${}^{\kappa}2$	${}^{\mathrm{cof}(\kappa)}\kappa$

To ensure that such spaces have weight κ, we always assume that

$$2^{<\kappa} = \kappa,$$

where $2^{<\kappa} = \sup_{\mu<\kappa} 2^{\mu}$. When κ is regular, this is equivalent to $\kappa^{<\kappa} = \kappa$. When κ is singular, this is equivalent to κ being strong limit.

Remark: We are also isolating the "right" notion(s) of Polish-like spaces in the generalized context (joint with Agostini and Schlicht).

Generalized Descriptive Set Theory

Definable sets: – *κ^+-Borel* sets $\rightsquigarrow$ κ^+-algebra generated by the topology $\boxed{\mathbf{Bor}(\kappa)}$

– *κ-analytic* sets $\rightsquigarrow$ continuous images of *closed subsets of* ${}^{\mathrm{cof}(\kappa)}\kappa$ $\boxed{\mathbf{\Sigma}^1_1(\kappa)}$

– *κ-coanalytic* sets $\rightsquigarrow$ complements of κ-analytic sets $\boxed{\mathbf{\Pi}^1_1(\kappa)}$

– *κ-bianalytic* sets $\rightsquigarrow$ sets which are both κ analytic and κ-coanalytic $\boxed{\mathbf{\Delta}^1_1(\kappa)}$

Stratification of $\mathbf{Bor}(\kappa)$: For $1 \leq \xi < \kappa^+$ we define

$\mathbf{\Sigma}^0_1(\kappa) =$ open sets $\qquad$ $\mathbf{\Pi}^0_1(\kappa) =$ closed sets

$\mathbf{\Sigma}^0_\xi(\kappa) =$ κ-sized unions of sets in previous levels $\qquad$ $\mathbf{\Pi}^0_\xi(\kappa) =$ κ-sized intersections of sets in previous levels

If $A \in \mathbf{Bor}(\kappa)$, its κ-Borel rank $\mathrm{rk}_B(A)$ is the smallest $1 \leq \xi < \kappa^+$ such that $A \in \mathbf{\Sigma}^0_\xi(\kappa) \cup \mathbf{\Pi}^0_\xi(\kappa)$; otherwise $\mathrm{rk}_B(A) = \infty$. We also set $\mathrm{rk}_B(A) = 0$ iff A is clopen.

Back to model theory...

Exactly as in the countable case, κ-sized models of a given first-order theory T can be coded as elements of ${}^{\kappa}2$ via characteristic functions.

$$\begin{aligned} \mathrm{Mod}^{\kappa}_{T} &\quad \rightsquigarrow \quad \text{(codes of) } \kappa\text{-sized models of } T \\ \cong^{\kappa}_{T} &\quad \rightsquigarrow \quad \text{isomorphism relation on } \mathrm{Mod}^{\kappa}_{T} \end{aligned}$$

Key point: When equipped with the induced topology $\mathrm{Mod}^{\kappa}_{T} \subseteq {}^{\kappa}2$ is a *κ-Borel* subset of ${}^{\kappa}2$, and $\cong^{\kappa}_{T}$ is a *κ-analytic* equivalence relation on it.

So we can study again the (G)DST-complexity of $\cong^{\kappa}_{T}$: κ-Borel vs true κ-analytic, κ-Borel rank, and so on.

Let's start with categoricity.

Mangraviti-M.

Assume that $2^{<\kappa} = \kappa \geq \aleph_0$ and κ is regular. Let T be a (non necessarily countable) complete theory in a signature of size $\leq \kappa$. TFAE:

1. T is κ-categorical;
2. $\cong^\kappa_T$ is open;
3. $\mathrm{rk}_B(\cong^\kappa_T) = 0$;
4. there is $\mathcal{M} \in \mathrm{Mod}^\kappa_T$ such that $[\mathcal{M}]_\cong$ is clopen.

In particular, there is no complete non-κ-categorical theory T for which $\cong^\kappa_T$ is a nontrivial open set.

In the countable case the result is even stronger.

Mangraviti-M.

Let T be a complete theory in a countable signature. TFAE:

1. T is countably categorical;
2. $\cong^\omega_T$ is clopen;
3. $\mathrm{rk}_B(\cong^\omega_T) \leq 1$, i.e. $\cong^\omega_T$ is open or closed;
4. $[\mathcal{M}]_\cong$ is open or closed for every $\mathcal{M} \in \mathrm{Mod}^\omega_T$;
5. there is $\mathcal{M} \in \mathrm{Mod}^\omega_T$ such that $[\mathcal{M}]_\cong$ is open or closed.

In particular, there is no complete theory T for which $\cong^\omega_T$ is a true open or true closed set.

Intermezzo: an application to (topological) classification theory

Definition

An equivalence relation on a topological space X is called **topologically smooth** if there is a Hausdorff space Y and a continuous $f\colon X \to Y$ such that for all $x_0, x_1 \in X$

$$x_0 \mathrel{E} x_1 \iff f(x_0) = f(x_1).$$

Theorem (Mangraviti-M.)

Let T be a complete first-order theory in a countable language. Then $\cong^\omega_T$ is topologically smooth if and only if T is countably categorical.

Therefore, unless T has just one countable model

> *it is not possible to classify in a continuous way the countable models of T up to isomorphism using invariants from a Hausdorff space.*

In particular, the quotient topology on $\mathrm{Mod}^\omega_T /\cong$ is never Hausdorff if the quotient space has more than one point.

A descriptive view of (old style) stability theory

Let T be again a countable complete first-order theory.

Shelah's Main Gap Theorem

Let $\kappa \geq \aleph_1$ be the γ-th cardinal.

1. If T is classifiable shallow of depth α, then $I(\kappa, T) \leq \beth_\alpha \left(|\gamma|^{2^{\aleph_0}} \right)$.
2. If T is not classifiable shallow, then $I(\kappa, T) = 2^\kappa$.

Topological spectrum function: $B(\kappa, T) = \mathrm{rk}_B(\cong^\kappa_T)$ (= the κ-Borel rank of $\cong^\kappa_T$)

Descriptive M.G.T. (Mangraviti-M., building on work by Friedman-Hyttinen-Kulikov)

Let κ be such that $\kappa^{<\kappa} = \kappa > 2^{\aleph_0}$.

1. If T is classifiable shallow of depth α, then $\cong^\kappa_T \in \mathbf{\Pi}^0_{4\alpha}$, hence $B(\kappa, T) \leq 4\alpha$.
2. If T is not classifiable shallow, then $\cong^\kappa_T$ is not even κ^+-Borel, i.e. $B(\kappa, T) = \infty$.

A descriptive view of (old style) stability theory

T countable complete first-order theory; $\kappa^{<\kappa} = \kappa = \aleph_\gamma > 2^{\aleph_0}$.

Uncountably categorical	Classifiable shallow $\mathrm{dp}(T) = \alpha$ (*)	Not classifiable shallow
$I(\kappa, T) = 1$	$I(\kappa, T) \leq \beth_\alpha\left(\|\gamma\|^{2^{\aleph_0}}\right)$	$I(\kappa, T) = 2^\kappa$
$B(\kappa, T) = 0$, i.e. (cl)open	$B(\kappa, T) \leq 4\alpha$	$B(\kappa, T) = \infty$, i.e. not κ^+-Borel

(*) Recall that $\alpha = \mathrm{dp}(T) < \aleph_1$, while the length of the κ^+-Borel hierarchy is $\kappa^+ \geq \aleph_3$.

In particular, there is no theory T such that $\cong^\kappa_T$ has uncountable κ^+-Borel rank (no matter which κ we look at).

A descriptive view of (old style) stability theory

Some remarks:

- When T is classifiable shallow, the function $B(\kappa, T)$ needs not to be non-decreasing in the first coordinate. However, it is *always* eventually bounded by a very small constant.
- There are forbidden complexity for $\cong^\kappa_T$, for example:
  – the isomorphism relation $\cong^\kappa_T$ cannot be a true $\boldsymbol{\Sigma}^0_\xi(\kappa)$-set if ξ is limit;
  – if $\kappa^{<\kappa} = \kappa = \aleph_\gamma$ and $\beth_{\omega_1}(|\gamma|) \leq \kappa$, then $\cong^\kappa_T$ cannot be a true $\boldsymbol{\Sigma}^0_\xi(\kappa)$ set or a true $\boldsymbol{\Pi}^0_\xi(\kappa)$ set *for any* $\xi < \kappa^+$.
- Part of the previous DST-complexity results extend beyond the scope of countable complete first-order theories.

Corollary (Mangraviti-M.)

Let $\kappa^{<\kappa} = \kappa > 2^{\aleph_0}$. Let $\mathcal{L}$ be a countable signature and $\mathcal{C} \subseteq \mathrm{Mod}^\kappa_{\mathcal{L}}$ be any collection of (codes for) κ-sized $\mathcal{L}$-structures *closed under elementary equivalence*. Then either $B(\kappa, \mathcal{C}) = \mathrm{rk}_B(\cong^\kappa_{\mathcal{C}}) \leq \aleph_1$, or else $B(\kappa, \mathcal{C}) = \infty$ (i.e. $\cong^\kappa_{\mathcal{C}}$ is not κ-Borel).

The singular case

Historically, GDST concentrated on regular cardinals. More recently, we started working on a GDST for singular κ's (joint works with Agostini, Dimonte, Pitton, ...).

Quite surprisingly, when $\mathrm{cof}(\kappa) = \omega$ the theory is extremely well-connected to well-developed areas in topology (Stone's generalized Baire spaces and alike) and pure set theory (very large cardinals, like Woodin's I0). It turned out that

> When $\mathrm{cof}(\kappa) = \omega$ and $2^{<\kappa} = \kappa$, GDST is precisely the theory of complete metric spaces of density κ.

This obviously includes important mathematical objects such as non-separable Banach spaces and so on.

The singular case: countable cofinality

The techniques from classical DST do not work in the generalized context. The list of missing tools when $\mathrm{cof}(\kappa) = \omega$ includes a reasonable Baire category theory and κ-Borel determinacy.

This makes even more remarkable the fact that, although with different methods, one can actually reconstruct a **very large** part of the classical theory. Here is a partial list:

- Surjective universality of the generalized Baire space ${}^{\omega}\kappa$ (in various forms).
- Topological characterizations of ${}^{\kappa}2$ and ${}^{\omega}\kappa$.
- Generalized Cantor-Bendixson theorem.
- Structural properties for the levels of the κ-Borel hierarchy (reduction, separation, ...).
- Luzin's separation theorem and Souslin's theorem (and their variants).
- Well-behaved theory of standard κ-Borel spaces (e.g. selection theorem for $F(X)$).
- "Small section" uniformization results for κ-Borel sets.
- κ-coanalytic ranks and boundedness theorems.
- κ-PSP for κ-analytic sets and beyond (under large cardinal assumptions like I0(κ)).

Structural results for equivalence relations in non-separable spaces

κ-Polish space = completely metrizable topological space with weight $\leq \kappa$

Generalized Silver's Dichotomy (Dimonte-M.)

Assume that κ is the limit of a countable sequence of measurable cardinals. If E is a κ-coanalytic equivalence relation on a κ-Polish space X, then either E has $\leq \kappa$-many classes, or else there is a closed copy of ${}^{\kappa}2$ inside X consisting of pairwise E-inequivalent elements.

1. Prove the generalized version of the classical G_0-dichotomy. *(Easy... after B. Miller)*
2. Equip X^2 with the topology induced by (a form of) the diagonal Prikry forcing. Observe that such a topology, which is far from being κ-Polish, comes with a reasonably well-behaved Baire category theory. *(Nontrivial, but reasonable)*
3. Play simultaneously with the κ-Polish topology and the diagonal Prikry topology to derive the Silver's dichotomy from the G_0-dichotomy. *(Difficult!)*

Remark: This is reminiscent of when in the classical setting one combines the Polish topology and the Gandy-Harrington topology to obtain certain dichotomies.

Structural results for equivalence relations in non-separable spaces

Generalized Burgess' Trichotomy (Dimonte-M.)

Assume that κ is the limit of a countable sequence of measurable cardinals and E is a κ-analytic equivalence relation on a κ-Polish space X. Then either E has $\leq \kappa$-many classes, or it has exactly κ^+-many classes, or else there is a closed copy of ${}^{\kappa}2$ inside X consisting of pairwise E-inequivalent elements.

1. There is a κ-Borel κ^+-filtration of E, i.e. a decreasing and continuous family $(E_\alpha)_{\alpha<\kappa^+}$ of κ-Borel equivalence relations such that $E = \bigcap_{\alpha<\kappa^+} E_\alpha$. *(Standard)*
2. By the generalized Silver's dichotomy, each E_α has either $\leq \kappa$-many classes, or else there is a closed copy of ${}^{\kappa}2$ inside X consisting of pairwise E_α-inequivalent elements. If the second alternative holds for some $\alpha < \kappa^+$, then it holds for E as well.
3. If all E_α's have $\leq \kappa$-many classes but E has more than κ^+-many classes, then there is a closed copy of $B(\kappa) \approx {}^{\kappa}2$ consisting of pairwise E-inequivalent elements. *(Difficult!)*

A generalized Morley's theorem

Recall that for countable models we have

Theorem (Morley)

$I(\aleph_0, T) \leq \aleph_0$, or $I(\aleph_0, T) = \aleph_1$, or else $I(\aleph_0, T) = 2^{\aleph_0}$.

Theorem (Dimonte-M.)

Assume that κ is the limit of a countable sequence of measurable cardinals and T be any theory in a countable signature.

- Either $I(\kappa, T) \leq \kappa$, of $I(\kappa, T) = \kappa^+$, or else $I(\kappa, T) = 2^\kappa$.
- If moreover $\cong^\kappa_T$ is κ^+-Borel, then either $I(\kappa, T) \leq \kappa$, or else $I(\kappa, T) = 2^\kappa$.

Remark: The scope of applicability of the theorem is quite wide: for example, the theory T might be incomplete, or even a (complete or incomplete) theory in the infinitary logic $\mathcal{L}_{\kappa^+\omega}$.

Thank you for your attention!

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