The Cauchy-Riemann Equations on the Hartogs Triangles

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Outline

1. The $\overline{\partial}$-problem and Dolbeault cohomology groups
2. L^2 theory for $\overline{\partial}$ on domains in $\mathbb{C}^n$
3. Function theory and $\overline{\partial}$ on the Hartogs triangle
4. The Cauchy-Riemann Equations in Complex Projective Spaces
5. The $\overline{\partial}$ operator on Hartogs triangles in $\mathbb{CP}^2$

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Dolbeault cohomology

The $\overline{\partial}$-problem

Let Ω be a domain in $\mathbb{C}^n$ (or a complex manifold), $n \geq 2$. Given a C^∞-smooth (p,q)-form g such that $\overline{\partial} g = 0$, find a smooth $(p, q-1)$-form u such that

$$\overline{\partial} u = g. \tag{1}$$

Dolbeault Cohomology

$$H^{p,q}(\Omega) = \frac{\ker\{\overline{\partial} : \mathcal{C}^\infty_{p,q}(\Omega) \to \mathcal{C}^\infty_{p,q+1}(\Omega)\}}{\text{range}\{\overline{\partial} : \mathcal{C}^\infty_{p,q-1}(\Omega) \to \mathcal{C}^\infty_{p,q}(\Omega)\}} \quad (H^{p,q}(\overline{\Omega}))$$

- Obstruction to solving the $\overline{\partial}$-problem on Ω.
- Natural topology arising as quotients of Fréchet topologies on $\ker(\overline{\partial})$ and range$(\overline{\partial})$.
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L^2 closure of unbounded operators

Two ways to close an unbounded operator in L^2

- (1) The (weak) maximal closure of $\overline{\partial}$: Dom$(\overline{\partial}) \subsetneq L^2_{p,q}(\Omega)$ is the largest. Realize $\overline{\partial}$ as a closed densely defined (maximal) operator

$$\overline{\partial} : L^2_{p,q}(\Omega) \to L^2_{p,q+1}(\Omega).$$

The L^2-Dolbeault Coholomolgy is defined by

$$H^{p,q}_{L^2}(\Omega) = \frac{\ker\{\overline{\partial} : L^2_{p,q}(\Omega) \to L^2_{p,q+1}(\Omega)\}}{\text{range}\{\overline{\partial} : L^2_{p,q-1}(\Omega) \to L^2_{p,q}(\Omega)\}}$$

- (2) The (strong) minimal closure of $\overline{\partial}$: Let $\overline{\partial}_c$ be the (strong) minimal closed L^2 extension of $\overline{\partial}$.

$$\overline{\partial}_c : L^2_{p,q}(\Omega) \to L^2_{p,q+1}(\Omega).$$

By this we mean that $f \in$ Dom$(\overline{\partial}_c)$ if and only if there exists a sequence of compactly supported smooth forms f_ν such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$.

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Let Ω be a bounded domain in $\mathbb{C}^n$.

Another two ways to close an unbounded operator in L^2

- (3) The (strong) maximal closure of $\overline{\partial} : C^\infty_{p,q}(\overline{\Omega}) \to C^\infty_{p,q+1}(\overline{\Omega})$.
  Let $\overline{\partial}_s : L^2_{p,q}(\Omega) \to L^2_{p,q+1}(\Omega)$ be the strong maximal closed L^2 extension of $\overline{\partial}$ on smooth forms in the L^2-graph norm. We say that $f \in \mathrm{Dom}(\overline{\partial}_s)$ if and only if there exists a sequence of smooth forms $f_\nu \in C^\infty_{p,q}(\overline{\Omega})$ such that $f_\nu \to f$ and $\overline{\partial} f_\nu \to \overline{\partial} f$ in L^2.
- (4) Solving $\overline{\partial}$ with prescribed support: Let $\overline{\partial}_{\tilde{c}} : L^2_{p,q}(\Omega) \to L^2_{p,q+1}(\Omega)$ be the weak minimal closed L^2 extension in the sense that $f \in \mathrm{Dom}(\overline{\partial}_{\tilde{c}})$ if and only if $\overline{\partial} f = g$ in $\mathbb{C}^n$ as distributions with compact support in $\overline{\Omega}$ for some $g \in L^2_{p,q+1}(\Omega)$ when f and g are extended as zero outside Ω.

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The $\overline{\partial}$-Neumann problem

Let $\Box_{p,q}$ ($\overline{\partial}$-Laplacian) be the closed self-adjoint densely defined (unbounded) operator :

$$\Box_{p,q} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : L^2_{p,q}(\Omega) \to L^2_{p,q}(\Omega)$$

Suppose that the range $\Box_{p,q}$ closed. $L^2_{p,q}(\Omega) = \mathrm{Range}(\Box_{p,q}) \oplus \ker(\Box_{p,q})$.
($\iff$ range of $\overline{\partial}$ is a closed subspace in $L^2_{p,q}(\Omega)$ and $L^2_{p,q+1}(\Omega)$.)

$\mathcal{H}^{p,q}(\Omega) = \ker(\Box_{p,q})$, [the space of Harmonic (p,q)-forms]

Consequences of the closed range property of $\overline{\partial}$

- (Hodge Theorem) The space $H^{p,q}_{L^2}(\Omega)$ is isomorphic to the space of harmonic forms $\mathcal{H}^{p,q}(\Omega)$.
- The operator $\Box_{p,q}$ is invertible on $\mathcal{H}^{p,q}(\Omega)^\perp$ and its inverse is called the $\overline{\partial}$- Neumann operator $N_{p,q}$.
- The $\overline{\partial}$ problem can be solved with L^2-estimates: If $g \perp \ker(\overline{\partial}^*)$, then there is u such that $\overline{\partial}u = g$, and $\|u\|_{L^2} \le C\,\|g\|_{L^2}$.

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Weak Equals Strong and L^2 Serre Duality

- $\overline{\partial}$ and $\overline{\partial}_c$ are dual to each other. $\overline{\partial}_s$ and $\overline{\partial}_{\tilde{c}}$ are dual to each other.
- $\overline{\partial}$ (or $\overline{\partial}_s$) has closed range $\iff$ $\overline{\partial}_c$ (or $\overline{\partial}_{\tilde{c}}$) has closed range.

Weak and Strong Extensions (Friedrichs-Hörmander (1965))

If Ω has Lipschitz boundary, then $\overline{\partial} = \overline{\partial}_s$ and $\overline{\partial}_c = \overline{\partial}_{\tilde{c}}$.

L^2 Serre duality (Chakrabarti-S (2012) Laurent-S (2013))

- Let $\star : L^2_{p,q}(\Omega) \to L^2_{n-p,n-q}(\Omega)$ be the Hodge star operator. We have

$$\star\Box_{p,q} = \Box^c_{n-p,n-q}\star,$$

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Weak Equals Strong and L^2 Serre Duality

- $\overline{\partial}$ and $\overline{\partial}_c$ are dual to each other. $\overline{\partial}_s$ and $\overline{\partial}_{\tilde{c}}$ are dual to each other.
- $\overline{\partial}$ (or $\overline{\partial}_s$) has closed range $\iff$ $\overline{\partial}_c$ (or $\overline{\partial}_{\tilde{c}}$) has closed range.

Weak and Strong Extensions (Friedrichs-Hörmander (1965))

If Ω has Lipschitz boundary, then $\overline{\partial} = \overline{\partial}_s$ and $\overline{\partial}_c = \overline{\partial}_{\tilde{c}}$.

L^2 Serre duality (Chakrabarti-S (2012) Laurent-S (2013))

- Let $\star : L^2_{p,q}(\Omega) \to L^2_{n-p,n-q}(\Omega)$ be the Hodge star operator. We have

$$\star\Box_{p,q} = \Box^c_{n-p,n-q}\star,$$

where $\Box^c = \overline{\partial}_c\overline{\partial}_c^* + \overline{\partial}_c^*\overline{\partial}_c$.

- If $\overline{\partial}$ has closed range in $L^2_{p,q}(\Omega)$ and $L^2_{p,q+1}(\Omega)$ $\implies H^{p,q}_{L^2}(\Omega) \cong H^{n-p,n-q}_{c,L^2}(\Omega)$ (since $\mathcal{H}^{p,q}(\Omega) \cong \mathcal{H}^{n-p,n-q}_{c,L^2}(\Omega)$).

Hörmander 1965

$\Omega \subset\subset \mathbb{C}^n$ is bounded and pseudoconvex $\implies H^{p,q}_{L^2}(\Omega) = 0, \qquad q > 0.$

The converse is also true if Ω If $\Omega \subset\subset \mathbb{C}^n$ has Lipschitz boundary. $H^{p,q}_{L^2}(\Omega) = 0$, $q > 0 \implies \Omega$ is pseudoconvex. (Fu 2005 Hearing Pseudoconvexity).

Sobolev estimates and boundary regularity for $\bar\partial$ (Kohn 1963, 1974)

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then

$$\begin{cases} H^{p,q}_{W^s}(\Omega) = 0, & s > 0,\ q > 0 \\ H^{p,q}(\overline{\Omega}) = 0. & \end{cases}$$

L^2 theory for $\overline{\partial}$ on pseudoconvex domains in $\mathbf{C}^n$

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Non-closed range property for $\overline{\partial}$

Laurent-S (2013)

Let Ω be a bounded Lipschitz domain in $\mathbb{C}^2$ such that $\mathbb{C}^2 \setminus \overline{\Omega}$ is connected. Suppose that Ω is not pseudoconvex. Then $H^{0,1}_{L^2}(\Omega)$ is non-Hausdorff.

If $\bar{\partial}$ has closed range in $L^{0,1}(\Omega)$, By L^2 Serre duality,
$H^{0,1}_{L^2}(\Omega) \cong H^{2,1}_{L^2,\overline{\partial}_c}(\Omega) \cong H^{0,1}_{L^2,\overline{\partial}_c}(\Omega) = 0 \iff \Omega$ is pseudoconvex.

Corollary

Either $H^{0,1}_{L^2}(\Omega) = 0$ (and Ω is pseudoconvex) or $H^{0,1}_{L^2}(\Omega)$ is non-Hausdorff.

- Similar results also hold for $(0, n-1)$-forms in $\mathbb{C}^n$ when $n \geq 3$.
- Laufer (1975) Let Ω be a domain in $\mathbb{C}^n$ (or a Stein manifold). Then either $H^{0,1}(\Omega) = 0$ or $H^{0,1}(\Omega)$ is infinite dimensional.
- Trapani (1986) obtained similar results in $H^{0,1}(\Omega)$.

The Hartogs Triangle in $\mathbb{C}^2$

Let T be the Hartogs triangle in $\mathbb{C}^2$ defined by

$$T = \{(z, w) \in \mathbb{C}^2 \mid |z| < |w| < 1\}.$$

- $T \cong D \times D_*$ and T is pseudoconvex
- T is not Lipschitz at 0.
- $\overline{T}$ does not have a Stein neighborhood basis. (The Diederich-Fornaess worm domains are smooth pseudoconvex domains without Stein neighborhood basis.)
- Since T is pseudoconvex, we have $H^{0,1}_{L^2}(T) = 0$.
- $\overline{\partial}$ has closed range in $L^2_{0,1}(T)$ and the range is equal to $\mathrm{Ker}(\overline{\partial})$.
- Does $\overline{\partial}_s$ have closed range?
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Global regularity and irregularity

Global Irregularity (Sibony 1980)

There exists $g \in \mathcal{C}^{\infty}_{0,1}(\overline{T})$ with $\overline{\partial} g = 0$, there does not exist $u \in \mathcal{C}^{\infty}(\overline{T})$ such that $\overline{\partial} u = g$ and $H^{0,1}(\overline{T})$ is infinite dimensional.

In fact, $H^{0,1}(\overline{T})$ is non-Hausdorff (Laurent-S 2015).

Global regularity (Chaumat-Chollet 1991)

For each positive integer k and $0 < \alpha < 1$, there exists $u \in C^{k,\alpha}(T)$ with $\overline{\partial} u = f$ for any $\overline{\partial}$-closed $f \in C^{k,\alpha}(T)$.

$$H^{0,1}_{C^{k,\alpha}}(T) = 0.$$

Notice that

$$\cap_k C^{k,\alpha}(T) = C^{\infty}(\overline{T}).$$

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Outline

Sobolev Extendability on T

Sobolev Extension Theorem: (Calderon-Stein)

Let $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. Then Ω is an extension domain. For each $k \in \mathbb{N}$ and $1 \le p \le \infty$, there exists a bounded linear operator

$$\eta_k : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$$

such that $\eta_k f\big|_{\Omega} = f$ for all $f \in W^{k,p}(\Omega)$.

- Proved by Whitney's extension and regulariiztaion.

Theorem (Burchard-Flynn-Lu-S 2022 Math. Zeit.)

The Hartogs triangle T is a Sobolev extension domain.

The Hartogs triangle T is not Lipschitz.
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Uniform domains

(ϵ, δ) and Uniform Domains: Let Ω be a domain in $\mathbb{R}^n$. The domain Ω is called an (ϵ, δ) *domain* if for every $p_1,\ p_2 \in \Omega$ and $|p_1 - p_2| < \delta$, there exists a rectifiable curve $\gamma \in \Omega$ joining x and y such that

$$\ell(\gamma) \leq \frac{1}{\epsilon}|p_1 - p_2|$$

and

$$\text{dist}(p, b\Omega) \geq \frac{\epsilon|p - p_1||p - p_2|}{|p_1 - p_2|} \quad \text{for all } p \in \gamma.$$

where $\ell(\gamma)$ denotes the Euclidean length of γ and $\text{dist}(p, b\Omega)$ denotes the distance from p to $b\Omega$.
When $\delta = \infty$, Ω is called a *uniform domain.*

Lemma

The Hartogs triangle is is a uniform domain (with $\epsilon = 0.01$).

From a theorem by Jones (1981), it is an extension domain.

Weak and Strong Sobolev spaces

Sobolev spaces on T

Let $W^1(T)$ denote the Sobolev space of L^2-functions on T with weak first-order derivatives in L^2. Then the following statements hold:

1. *(Smooth approximation).* $C^\infty(\overline{T})$ is dense in $W^1(T)$.
2. *(Sobolev embedding).* $W^1(T) \subset L^4(T)$, and the inclusion map is bounded.
3. *(Rellich lemma).* The inclusion $W^1(T) \subset L^2(T)$ is compact.

Poincaré's inequality

There exists a constant $C > 0$ such that

$$\|f\|^2 \leq C\|df\|^2$$

for all $f \in W^1(T)$ with $(f, 1) = 0$, where $\| \ \|$ denotes the L^2-norm on T.

Applications and Open Questions

Applications of the Sobolev extension theorem

- The Hartogs triangle T is a chord-arc domain. The trace theorem holds.
- $d : L^2(T) \to L^2_1(T)$ has closed range and $d = d_s$ (Poincaré's Inequality holds).
- The Neumann boundary value problem is solvable.
  Given any $f \in L^2(T)$ such that $(f, 1) = \int_T f = 0$, there exists $u = G_\nu f \in W^1(T)$ such that

$$(du, d\phi) = (f, \phi) \qquad \text{for all } \phi \in W^1(T).$$

- The solution $G_\nu : L^2(T) \to L^2(T)$ is compact (by the Rellich lemma).

Open Questions:

- Does $d_q : L^2_q(T) \to L^2_{q+1}(T)$ have closed range? $q = 1, 2$.
- Does the Hodge theorem holds for $L^2_q(T)$?
  $\iff$ Does $\triangle_q = d_{q-1}d_q^* + d_{q+1}^* d_q$ have closed range?
- $q = 0$ or $q = 3$, Yes and $d = d_s$.

Weak equals strong for $\overline{\partial}$ (Burchard-Flynn-Lu-S.)

On T, we have $\overline{\partial} = \overline{\partial}_s$.

Proof:

- Let $\mathcal{H}(T) = \text{Ker}(\overline{\partial}) \cap L^2(T)$ be the Bergman space. Since $T \cong D \times D_*$, we can analyze $\mathcal{H}$ by Laurent expansions.
- We first show
$$\text{Ker}(\overline{\partial}) = \text{Ker}(\overline{\partial}_s) = \mathcal{H}.$$
- $\overline{\partial}_c = \overline{\partial}_{\tilde{c}}$ on functions using the Sobolev Embedding Theorem for T.
- From L^2 Serre duality, we have $\overline{\partial}_s : L^2_{0,1}(T) \to L^2_{0,2}(T)$ has closed range and $\text{Range}(\overline{\partial}_s) = L^2_{0,2}(T)$.
- $q = 1$, $\overline{\partial}_s : L^2(T) \to L^2_{0,1}(T)$ has closed range and
$$\text{Range}(\overline{\partial}_s) = \text{Range}(\overline{\partial}).$$

Cohomology on Annuli

Hörmander 2002, Shaw 2005

Let $\Omega = \Omega_1 \setminus \overline{\Omega}_2 \Subset \mathbb{C}^2$ where $\Omega_1 \Subset \Omega_2$ are bounded pseudoconves domains with smooth boundary. Then

$$H^{0,1}_{L^2}(\Omega) \cong \mathcal{H}(\Omega_2).$$

In particular, $H^{0,1}_{W^1}(\Omega)$ is Hausdorff and infinite dimensional.

Dolbeault cohomology on the complement of T

Suppose Ω_1 has C^2 boundary and $\Omega_2 = T$. Then

$$H^{0,1}_{W^1}(\Omega) \cong \mathcal{H}(T).$$

- (Dollar Bill Question) Is $H^{0,1}_{L^2}(\Omega)$ is Hausdorff?
- $\iff$ $H^{0,1}_{W^1}(\Omega_2) = 0$ ($H^{0,1}(\Omega)$ is non-Hausdorff) (Laurent-S 2013.)
- When $\Omega_2 = D^2$, $H^{0,1}_{L^2}(\Omega)$ is Hausdorff (Chakrabarti-Laurent-S2017).

Outline

Complex manifolds

Let X be a complex manifold. All the results for $\mathbb{C}^n$ can be extended to Stein manifolds (X is Stein $\iff$ X is a closed subspace of $\mathbb{C}^N$).
$\implies H^{p,q}(X) = 0,\ q \neq 0.$

- Let X be a compact complex manifold. Then $H^{p,q}(X)$ is finite-dimensional for all $0 \leq p, q \leq n$.
- Suppose that X is a complex manifold with strongly pseudoconvex boundary. Then $H^{p,q}(X)$ is finite-dimensional for all $q \neq 0$.
- There exists a complex manifold X (not Stein) with weakly pseudoconvex boundary such that $H^{0,1}(X)$ is not Hausdorff. (Malgrange (1975)).
- There exists a complex Stein manifold X with weakly pseudoconvex boundary such that $H^{2,1}_{L^2}(X)$ is not Hausdorff (but $H^{2,1}(X) = 0$). (Chakrabarti-S (2015)).

The $\overline{\partial}$-problem on pseudoonvex domains in $\mathbb{CP}^n$

Takeuchi (1964)

Let $\Omega \Subset \mathbb{CP}^n$ be a pseudoconvex domain. Then Ω is Stein (Hence $H^{p,q}(\Omega) = 0,\ q \neq 0.$)

The Bochner-Kodaira-Morrey-Kohn formula:

Let Ω be a domain with smooth boundary. For $u \in C^1_{p,q}(\overline{\Omega}) \cap \mathrm{Dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$,

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 = \|\overline{\nabla} u\|^2 + (\Theta_{p,q} u,\ u) + \int_{b\Omega} \langle (\partial\overline{\partial}\rho) u,\ u\rangle dS \tag{2}$$

where ρ is a defining function, dS is the induced surface element on $b\Omega$, $|\overline{\nabla} u|^2 = \sum_{j=1}^n |\nabla_{\bar{L}_j} u|^2$ and $\Theta_{p,q}$ is the curvature term associated with the Fubini-Study metric.

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L^2 theory for $\overline{\partial}$ in $\mathbb{CP}^n$

The curvature term

- $\langle \Theta u, u \rangle = q(2n+1)|u|^2 \quad \text{if} \quad p = 0;$
- $\langle \Theta u, u \rangle = 0, \quad \text{if} \quad p = n;$
- $\langle \Theta u, u \rangle \geq 0, \quad \text{if} \quad p \geq 1.$

Theorem

Let $\Omega \Subset \mathbb{CP}^n$ be a pseudoconvex domain. Then $H^{0,q}_{L^2}(\Omega) = 0$, $q > 0$.

Suppose Ω has smooth boundary. B-K-M-K formula gives

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 = \|\overline{\nabla} u\|^2 + (\Theta_{p,q} u,\ u) + \int_{b\Omega} \langle (\partial\overline{\partial}\rho) u,\ u \rangle dS$$
$$\geq q(2n+1)\|u\|^2.$$

This gives an alternative proof to the Hörmander's L^2 theorem.

L^2 theory and Boundary Regularity

Bounded plurisubharmonic exhaustion functions

Let $\Omega \Subset \mathbb{CP}^n$ with Lipschitz boundary. There exist a distance function δ and an $0 < \eta \leq 1$ such that

$$i\partial\overline{\partial}(-\delta^\eta) > 0.$$

There exists a bounded plurisubharmonic exhaustion function for Ω. (Ohsawa-Sibony) for C^2 boundary (1998) and (Harrington) for Lipschitz boundary (2017). η is called the Diederich-Fornaess exponent.

L^2 Existence and Boundary Regularity (Berndtsson-Charpentier 2000)

- $H^{p,q}_{L^2}(\Omega) = 0$ for all $q > 0$
- Let η be the Diederich-Fornaess exponent with $0 < \eta \leq 1$.
- $B : W^\epsilon(\Omega) \to W^\epsilon(\Omega)$, $\epsilon < \frac{\eta}{2}$.
  $N : W^\epsilon(\Omega) \to W^\epsilon(\Omega)$ (Cao-S-Wang 2004).

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Outline

Hartogs' Triangles in $\mathbb{CP}^2$

In $\mathbb{CP}^2$, we denote the homogeneous coordinates by $[z_0, z_1, z_2]$. On the domain where $z_0 \neq 0$, we set $z = \frac{z_1}{z_0}$ and $w = \frac{z_2}{z_0}$.
Let H^+ and H^- be defined by

$$H^+ = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| < |z_2|\}$$
$$H^- = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| > |z_2|\}$$

$$M = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| = |z_2|\}.$$

$$H^+ \cup M \cup H^- = \mathbb{CP}^2.$$

These domains are called Hartogs' triangles in $\mathbb{CP}^2$. It is not Lipschitz near 0.

Complex foliation in $\mathbb{CP}^2$

The boundary

$$\begin{aligned} M &= \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| = |z_2|,\ z_0 \neq 0\} \\ &= \{(z, w) \in \mathbb{C}^2 \mid |z| = |w|\}. \end{aligned}$$

- For each $\theta \in \mathbb{R}$,

$$S_\theta = \{(z, w) \in \mathbb{C}^2 \mid z = e^{i\theta} w\}.$$

- Each S_θ is a compact Riemann surface.
- $M = \cup_\theta S_\theta$.
- $\cap_\theta S_\theta = 0$. M is not foliated by complex curves at 0.

L^2 theory for $\overline{\partial}_s$ on Hartogs Triangles

- Both H^+ and H^- are pseudoconvex ($\cong \mathbb{C} \times D$).
- M is a (non-Lipschitz) Levi-flat hypersurface in $\mathbb{CP}^2$ in the sense that M splits $\mathbb{CP}^2$ into two pseudoconvex domains.

Theorem (Laurent-S, 2018, Trans. AMS)

We have that $H^{2,1}_{\overline{\partial}_s, L^2}(H^+)$ is infinite dimensional.

Sketch of proof:

- By analyzing the Bergman space $\mathcal{H}(H^-)$, we have $\mathcal{H}(H^-) \cap W^1(H^-)$ is infinite dimensional. Then $H^{2,1}_{\overline{\partial}_{\tilde{c}}, L^2}(H^+)$ is infinite dimensional.
- Suppose that $\overline{\partial}_s$ has closed range, then $\overline{\partial}_{\tilde{c}}$ has closed range.
- L^2 Serre duality $\implies H^{2,1}_{\overline{\partial}_s, L^2}(H^+) \cong H^{2,1}_{\overline{\partial}_{\tilde{c}}, L^2}(H^+)$, which is infinite dimensional.

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- Both H^+ and H^- are pseudoconvex ($\cong \mathbb{C} \times D$).
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Further results and open questions

The Hartogs triangle and Levi-flat hypersurfaces

- $\overline{\partial} = \overline{\partial}_s$ on $H^{\pm} \implies H^{2,1}_{L^2}(H^+)$ is infinitely dimensional.
- Is $H^{2,1}_{L^2}(H^+)$ Hausdorff? ($\iff$ Does $\overline{\partial}$ have closed range in $L^2_{2,1}(H^+)$?)
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Boundary regularity

For a pseudoconvex domain $\Omega \Subset \mathbb{CP}^n$ with smooth boundary, do we have

- $H^{0,1}_{W^1}(\Omega) = 0$? (We do have $H^{0,1}_{L^2}(\Omega) = 0$.)
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Thank You