Generalized model theory and continuous logic

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Model Theoretic Logics and their Frontiers

Logic S_∞-space

Let $L = (R_i^{n_i})_{i \in I}$ be a countable relational language and

$$\mathcal{X}_L = \Pi_{i \in I} 2^{\omega^{n_i}}$$

be the corresponding space under the product topology τ.

$\mathcal{X}_L$ is the space of all L-structures on ω:
$\mathrm{x} = (\ldots x_i \ldots) \in \mathcal{X}_L \Leftarrow$ structure $(\omega, R_i)_{i \in I}$,
R_i is the n_i-ary relation defined by $x_i : \omega^{n_i} \to 2$.

The **logic action** of S_∞ is defined on $\mathcal{X}_L$ by the rule:

$$g \circ \mathrm{x} = \mathrm{y} \Leftrightarrow \forall i \forall \bar{s} (y_i(\bar{s}) = x_i(g^{-1}(\bar{s}))).$$

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Other topologies

For any countable fragment F of $L_{\omega_1\omega}$, which is closed under quantifiers, all sets

$$Mod(\phi, \bar{s}) = \{M \in \mathcal{X}_L : M \models \phi(\bar{s})\} \text{ with } \bar{s} \subset \omega$$

form a basis defining another topology (denoted by t_F) of the S_∞-space $\mathcal{X}_L$.

The logic action of the group S_∞ on $\mathcal{X}_L$ is continuous with respect to t_F.

Space of expansions

Let $G \leq_{closed} S_\infty$.

When $M_0 = (\omega, \ldots)$ with $G = \mathrm{Aut}(M_0)$ then a topology similar to τ can be defined on the G-space of all L-expansions of M_0.

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General case when $G \leq_{closed} S_\infty$

Fix $G \leq_{closed} S_\infty$ and
$(\langle \mathcal{X}, \tau \rangle, G)$ = Polish G-space with a countable basis.

Along with τ we shall consider another topology on $\mathcal{X}$.

Nice topology:

(below $\mathcal{N}^G$ = standard basis of the topology of G)

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Nice topology

Definition (H.Becker) A topology **t** on $\mathcal{X}$ is **nice** for the G-space $(\langle \mathcal{X}, \tau \rangle, G)$ if:
(A) **t** is a Polish, **t** is finer than τ and the G-action remains **t**-continuous.
(B) There exists a basis $\mathcal{B}$ for **t** (called **nice**) such that:

1. $\mathcal{B}$ is countable;
2. for all $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{B}$;
3. for all $B \in \mathcal{B}$, $\mathcal{X} \setminus B \in \mathcal{B}$;
4. for all $B \in \mathcal{B}$ and $u \in \mathcal{N}^G$, $B^{\Delta u}, B^{\star u} \in \mathcal{B}$;
5. for any $B \in \mathcal{B}$ there exists an open subgroup $H < G$ such that B is invariant under the corresponding H-action.

Logic space for Polish groups?

Question:
Is it possible to extend the generalised model theory of H.Becker to actions of Polish groups (without the assumption $G \leq S_\infty$) ?

Looking for terminology. Canonical structure for G

Let (G, d) be a Polish group with a left invariant metric ≤ 1.
If $(\mathcal{X}, d)$ is its completion, then $G \leq Iso(\mathcal{X})$.

J.Melleray: Any Polish G *is the automorphism group of the continuous structure on* $\mathcal{X}$, say M_G.

Let $S \subseteq_{cntble,dnse} \mathcal{X}$. Enumerate all orbits of G of finite tuples of S.

For the closure of such an n-orbit C define a predicate $R_{\overline{C}}$ on $(\mathcal{X}, d)$ (with continuity moduli $= id$) by

$$R_{\overline{C}}(y_1, ..., y_n) = d((y_1, ..., y_n), \overline{C}) \text{ (i.e. } inf\{d(\bar{y}, \bar{c}) : \bar{c} \in \overline{C}\}).$$

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The space of continuous structures

Fix a continuous signature L and Polish $(\mathcal{Y}, d)$;
S be a dense cntble $\subseteq \mathcal{Y}$.

- The **Polish space** $\mathcal{Y}_L$ *of continuous L-strctres on* $(\mathcal{Y}, d)$:

Metric: Enumerate all $(j, \bar{s})$, where $\bar{s} \in S$ and $|\bar{s}| = arity(R_j)$.
For L-structures M and N on $\mathcal{Y}$ let

$$\delta(M,N) = \sum_{i=1}^{\infty} \{2^{-i} |R_j^M(\bar{s}) - R_j^N(\bar{s})| \ : \ i \text{ is the number of } (j, \bar{s})\}.$$

Logic action: the Polish group $Iso(\mathcal{Y})$ acts on $\mathcal{Y}_L$ continuously

Taking $\mathcal{Y} = M_G$ we get a G-space of L-expansions of M_G.

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Universality

Theorem ([CL], [IMI])
For any Polish group G there is Polish $(\mathcal{Y}, d)$ and a continuous relational signature L such that

- $G < Iso(\mathcal{Y})$
- for any Polish $(G, \mathcal{X})$ there is a Borel 1-1-map $\mathcal{M} : \mathcal{X} \to \mathcal{Y}_L$ s. t. for any $x, x' \in \mathcal{X}$ structures $\mathcal{M}(x)$ and $\mathcal{M}(x')$ are isomorphic if and only if x and x' are in the same G-orbit,

The map $\mathcal{M}$ is a Borel G-invariant 1-1-reduction of $(\mathcal{X}, E_G)$ to $(\mathcal{Y}_L, E_{Iso(\mathcal{Y})})$.

Looking for terminology

Find counterparts for Mod$(\phi, \bar{s})$ *and* $\bar{s}$*-stabilizers.*

Grey subsets and subgroups

A **grey subset** of $\mathcal{X}$, denoted $\phi \sqsubseteq \mathcal{X}$, is a function $\mathcal{X} \to [0,1]$.

It is **open (closed)**, $\phi \in \Sigma_1$ (resp. $\phi \in \Pi_1$), if the **cone** $\phi_{<r}$ (resp. $\phi_{>r}$) is open for all $r \in [0,1]$

(here $\phi_{<r} = \{z \in \mathcal{X} : \phi(z) < r\}$).

(We also define Borel classes Σ_α, Π_α).

When G is a Polish group, then $H \sqsubseteq G$ is called a **grey subgroup** if $H(1) = 0$, $\forall g \in G(H(g) = H(g^{-1}))$ and $\forall g, g' \in G(H(gg') \leq H(g) + H(g'))$.

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Grey stabilizer

Basic example:
For $\bar{c}$ from $(\mathcal{Y}, d)$ and a linear δ with $\delta(0) = 0$
grey stabilizer $H_{\delta,\bar{c}} \sqsubseteq Iso(\mathcal{Y})$:

$$H_{\delta,\bar{c}}(g) = \delta((d(\bar{c}, g(\bar{c}))), \text{ where } g \in Iso(\mathcal{Y}).$$

Example: grey subsets of $\mathcal{Y}_L$

A **continuous formula** is an expression built from 0,1 and atomic formulas by applications of the following functions:

$$x/2 \ , \ x \dot{-} y = max(x-y,0) \ , \ min(x,y) \ , \ \ldots \ , \ sup_x \text{ and } inf_x .$$

Any continuous sentence $\phi(\bar{c})$ defines a grey subset of $\mathcal{Y}_L$ which belongs to Σ_n for some n:

$$\phi(\bar{c}) \text{ takes } M \text{ to the value } \phi^M(\bar{c}).$$

Invariant grey subsets

Definition $\mathcal{X} = G$-space.
A grey $\phi \sqsubseteq \mathcal{X}$ is **invariant** with respect to $H \sqsubseteq G$ if for any $g \in G$ we have $\phi(g(x)) \leq \phi(x) \dot{+} H(g)$.

Example: Assuming that continuity moduli of L-symbols are id for any continuous $\phi(\bar{x})$ there is a linear function δ such that

$$H_{\delta,\bar{c}}(g) = \delta((d(\bar{c}, g(\bar{c}))), \text{ where } g \in Iso(\mathrm{Y}).$$

and the grey subset $\phi(\bar{c}) \sqsubseteq \mathcal{Y}_L$ satisfy

$$\phi^{g(M)}(\bar{c}) \leq \phi^M(\bar{c}) \dot{+} H_{\delta,\bar{c}}(g).$$

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Grey bases

1. Distinguish $\mathcal{R}$, a countable family of open grey $\sqsubseteq G$ so that
   - all $\rho_{<r}$ for $\rho \in \mathcal{R}$ and $r \in \mathbb{Q}$, form a basis of the topology of G.
   - $\mathcal{R}$ consists of **grey cosets**, i.e. for such $\rho \in \mathcal{R}$ there is a grey subgroup $H \in \mathcal{R}$ and an element $g_0 \in G$ so that for any $g \in G$, $\rho(g) = H(gg_0^{-1})$.

2. Considering a $(G, \mathcal{R})$-space $\mathcal{X}$ we distinguish a cntble family $\mathcal{U}$ of open grey sbsts of $\mathcal{X}$ generating the topol.

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Nice basis

Definition. A family $\mathcal{B}$ of Borel grey subsets of the G-space $\mathcal{X}$ is a **nice basis** w.r.to $\mathcal{R}$ if:

- $\mathcal{B}$ is countable and generates the topol. finer than τ;
- for all $\phi_1, \phi_2 \in \mathcal{B}$, the functions $min(\phi_1, \phi_2)$, $max(\phi_1, \phi_2)$, $|\phi_1 - \phi_2|$, $\phi_1 \dot{-} \phi_2$ $\phi_1 \dot{+} \phi_2$ belong to $\mathcal{B}$;
- for all $\phi \in \mathcal{B}$ and rational $r \in [0, 1]$, $r\phi$ and $1 - \phi \in \mathcal{B}$;
- for all $\phi \in \mathcal{B}$ and $\rho \in \mathcal{R}$, $\phi^{*\rho}, \phi^{\Delta\rho} \in \mathcal{B}$;
- any $\phi \in \mathcal{B}$ is invariant w.r.to some open grey subgrp $H \in \mathcal{R}$.

A topology $\mathbf{t}$ on $\mathcal{X}$ is $\mathcal{R}$-**nice** for the G-space $\langle \mathcal{X}, \tau \rangle$ if:
(a) $\mathbf{t}$ is Polish, $\mathbf{t}$ is finer than τ and $(G, \mathcal{X})$ is continuous w.r.to $\mathbf{t}$;
(b) there exists a nice basis $\mathcal{B}$ so that $\mathbf{t}$ is generated by all $\phi_{<q}$ with $\phi \in \mathcal{B}$ and $q \in \mathbb{Q} \cap (0, 1]$.

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The Urysohn sphere

The Urysohn sphere $\mathfrak{U}$ is the unique Polish metric space of diameter 1 which is universal and ultrahomogeneous.

The *rational Urysohn sphere*, $\mathbb{Q}\mathfrak{U}$, is both ultrahomogeneous and universal for countable metric spaces with rational distances and diameter ≤ 1.

There is a nice embedding of $\mathbb{Q}\mathfrak{U}$ into $\mathfrak{U}$.
Let G_0 be a dense countable subgroup of $\mathrm{Iso}(\mathbb{Q}\mathfrak{U})$; we may view it as a subgroup of $\mathrm{Iso}(\mathfrak{U})$.

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The Urysohn sphere as a platform. $\mathcal{R}^{\mathfrak{U}}(G_0)$

Let $\mathcal{R}_0$ be the family of all clopen grey subgroups of $\mathrm{Aut}(\mathfrak{U})$ of the (truncated) form

$$H_{q,\bar{s}} : g \to q \cdot d(g(\bar{s}), \bar{s}), \text{ where } \bar{s} \subset \mathbb{Q}\mathfrak{U}, \text{ and } q \in \mathbb{Q}^{+}.$$

($\mathcal{R}_0$ is closed under conjugacy by elements of G_0)

Consider the closure of $\mathcal{R}_0$ under the function **max** and define $\mathcal{R}^{\mathfrak{U}}(G_0)$ to be the family of all G_0-cosets of grey subgroups from $\max(\mathcal{R}_0)$.

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The Urysohn space as a platform. $\mathcal{B}_{\mathcal{L}}$

Let $\mathcal{L}$ be a countable fragment of $L_{\omega_1\omega}$ and

let $\mathcal{B}_{\mathcal{L}}$ be the family of all grey subsets of $\mathfrak{U}_{\mathcal{L}}$ defined by continuous $\mathcal{L}$-sentences (with parameters from $\mathbb{Q}\mathfrak{U}$) as above.

The space $\mathfrak{U}_L$

Theorem (IMI17)

The family $\mathcal{B}_{\mathcal{L}}$ is a $\mathcal{R}^{\mathfrak{U}}(G_0)$-nice basis.

Similar constructions (with weaker forms of this theorem, where **nice** *is replaced by* **good***):*

- The complex Hilbert space $l_2(\mathbb{N})$.
- The measure algebra on $[0,1]$.

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Existence

Theorem ([IMI17]).
Let $(G, \mathcal{R})$ be a Polish group with $\mathcal{R}$ satisfying
(i) for every grey subgroup $H \in \mathcal{R}$ if $gH \in \mathcal{R}$, then $H^g \in \mathcal{R}$;
(ii) $\mathcal{R}$ is closed under **max** and multiplying by rationals.
Let $\langle \mathcal{X}, \tau \rangle$ be a G-space and
$\mathcal{U}$ be a countable family of Borel grey subsets of $\mathcal{X}$ generating a topology finer than τ, so that
each $\phi \in \mathcal{U}$ is invariant w.r.to some grey subgroup $H \in \mathcal{R}$.

Then there is an $\mathcal{R}$-nice topology for $(\langle \mathcal{X}, \tau \rangle, G)$ so that $\mathcal{U}$ consists of open grey subsets.

Ubiquity in the case of $\mathfrak{U}$

Theorem ([IMI17]). Let $G = \mathrm{Iso}(\mathfrak{U})$.
Consider the logic G-space $\mathfrak{U}_L$ under the standard topology τ.
Let $\mathcal{F}$ be a countable family of Borel grey subsets of $\mathfrak{U}_L$ generating a topology finer than τ such that any $\phi \in \mathcal{F}$ is invariant w.r.to a grey subgroup $H \in \mathcal{R}^{\mathfrak{U}}$.

Then there is an $\mathcal{R}^{\mathfrak{U}}$-nice topology $\mathbf{t}$ for the G-space $\langle \mathfrak{U}_L, \tau \rangle$ which is generated by some countable fragment of $L_{\omega_1 \omega}$ such that $\mathcal{F}$ consists of $\mathbf{t}$-open grey subsets.

Lindström

G is a Polish group with a grey basis $\mathcal{R}$ consisting of grey cosets, $\langle \mathcal{X}, \tau \rangle$ is a Polish G-space, ect.

Theorem

Let $\mathbf{t}$ *be* $\mathcal{R}$*-good.*
Let $Y = Gx_0$ *for some (any)* $x_0 \in Y$ *and* Y *be a* G_δ*-subset of* $\mathcal{X}$*.*
Then both topologies τ *and* $\mathbf{t}$ *are equal on* Y*.*

Companions

Let X_0 and X_1 be closed invariant subsets of $(\mathcal{X}, \mathbf{t})$.
X_1 is a **companion** of X_0 if τ-closures of X_0 and X_1 coincide and any element of $\mathcal{B}$ is τ-clopen on X_1.

Effros space

Given a Polish space $\mathcal{Y}$ the **Effros structure** on $\mathcal{F}(\mathcal{Y})$ is the Borel space with respect to the σ-algebra generated by

$$\mathcal{C}_U = \{D \in \mathcal{F}(\mathcal{Y}) : D \cap U \neq \emptyset\},$$

for open $U \subseteq \mathcal{Y}$.

Given a Polish group G and a continuous (or Borel) action $(G\ \mathcal{Y})$, grey basis $\mathcal{R}$ (for G) and $\mathcal{B}$ (for $\mathbf{t}$ on $\mathcal{Y}$) consider $\mathcal{F}((\mathcal{Y}, \mathrm{t}))^m \times \mathcal{F}(G)^n$.

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For example

Coskey and Lupini:
For any Polish G and any standard Borel G-space $\mathcal{X}$ there is a continuous group monomorphism $\Phi : G \to \mathrm{Iso}(\mathfrak{U})$ and a Borel Φ-equivariant injection $f : \mathcal{X} \to \mathfrak{U}_L$.

All Polish groups can be considered as elements of $\mathcal{F}(\mathrm{Iso}(\mathfrak{U}))$, all Polish spaces are elements of $\mathcal{F}(\mathfrak{U}_L)$ and Polish G-spaces are pairs from $\mathcal{F}(\mathfrak{U}_L) \times \mathcal{F}(\mathrm{Iso}(\mathfrak{U}))$.

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Complete theories

G is a Polish group with a grey basis $\mathcal{R}$ consisting of grey cosets, $(\mathcal{X}, \tau)$, $\mathbf{t}$, $\mathcal{B}$, ...

Observation. The set of indecomposable G-invariant members $X \in \mathcal{F}(\mathcal{X}, \mathbf{t})$ (i.e. "complete theories") is Borel.

Complexity of companions

Fix an enumeration of the sets

$$\mathcal{B}(\mathbb{Q}) = \{(\phi)_{<r} : \phi \in \mathcal{B} \text{ and } r \in \mathbb{Q} \cap [0,1]\},$$

$$\mathcal{B}_o(\mathbb{Q}) = \{(\phi)_{<r} : \phi \text{ is a clopen member of } \mathcal{B} \text{ and } r \in \mathbb{Q} \cap [0,1]\}$$

(a basis of the topology τ)

Theorem. The set of pairs (X_0, X_1) of G-invariant members of $\mathcal{F}(\mathcal{X}, \mathbf{t})$ with the condition that X_1 is a companion of X_0 is Borel.

A Borel substitute for stability

Given a $(G,\mathcal{R})$-space $\mathcal{X}$, a nice basis $\mathcal{B}$, a grey subset $\phi \in \mathcal{B}$ and a **t**-closed subset $Y \subseteq \mathcal{X}$ define the notion ϕ is **unstable** w.r. to Y. (ϕ tgether wth some $H, H' \in \mathcal{R}$ s.t. ϕ is invrnt w.r to $\max(H, H')$).

Example: $\mathfrak{U}$, $\mathbb{Q}\mathfrak{U}$, G_0 (dense cntable $\leq \mathrm{Iso}(\mathbb{Q}\mathfrak{U})$) , $\mathcal{R}^{\mathfrak{U}}(G_0)$, $\mathcal{B}_{\mathcal{L}}$.

$\phi(\bar{s}\bar{s}') \in \mathcal{B}_{\mathcal{L}}$ is *unstable* w.r. to Y if for any n and any ε
$\exists \bar{s}_1\bar{s}'_1 \dots \bar{s}_n\bar{s}'_n$ s.t. $tp^{\mathfrak{U}}(\bar{s}_i\bar{s}'_j) =_{\varepsilon} tp^{\mathfrak{U}}(\bar{s}\bar{s}')$ for all i, j and
$Y \cap \bigcap\{(\phi(\bar{s}_i\bar{s}'_j))_{\leq\varepsilon} : i < j\} \cap \bigcap\{(\phi(\bar{s}_i\bar{s}'_j))_{\geq 1-\varepsilon} : j \leq i\} \neq \emptyset$.

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Stable platforms

The platform $\mathfrak{U}$ is not stable.
The following platforms are stable:

- The complex Hilbert space $l_2(\mathbb{N})$.
- The Polish ultrametric Urysohn space for $\mathbb{Q} \cap [0,1]$.

A Borel substitute for extremely amenable theories

Given a $(G, \mathcal{R})$-space $\mathcal{X}$, $\mathcal{B}$, $H \in \mathcal{R}$, $\phi \in \mathcal{B}$ and a **t**-closed subset $Y \subseteq \mathcal{X}$ define the notion

$((\phi)_{\leq 0}, H)$ **extends to an invariant type** of Y.

Example: $\mathfrak{U}$, $\mathbb{Q}\mathfrak{U}$, G_0 (dense cntable $\leq$ Iso($\mathbb{Q}\mathfrak{U}$)) , $\mathcal{R}^{\mathfrak{U}}(G_0)$, $\mathcal{B}_{\mathcal{L}}$.

$\phi(\bar{s}_0\bar{s}')$ and $H_{1,\bar{s}_0}$ extend to an *invariant type* of Y
if for any ε and any $\bar{s}_1 \ldots \bar{s}_n$ s.t. $tp^{\mathfrak{U}}(\bar{s}_0) =_{\varepsilon} tp^{\mathfrak{U}}(\bar{s}_i)$ for all i
there exists $\bar{s}$ s.t. $Y \cap \bigcap \{(\phi(\bar{s}_i\bar{s}))_{\leq \varepsilon} : i \leq n\} \neq \emptyset$.

A substitute for "Y **is e.a.**": any $((\phi)_{\leq 0}, H)$ extends to an invariant type of Y.

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Complexity and e.a.

Theorem. The set of X, G-invariant members of $\mathcal{F}(\mathcal{X}\ ,\mathbf{t})$, with the condition that any $((\phi)_{\leq 0}, H)$ extends to an invariant type of X is Borel.

Paper

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