On minimal expansions of elementary extensions of the group of integers

Itay Kaplan, January 14, 2022, Model Theoretic Logics and their Frontiers online conference

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Reducts and expansions

Definition 1. Suppose that $\mathcal{A}$ and $\mathcal{C}$ are two structures with the same universe.

(1) The structure $\mathcal{A}$ is a *reduct* of $\mathcal{C}$ if whenever $X \subseteq \mathcal{C}^k$ is definable in $\mathcal{A}$ (where definable means with parameters), it is also definable in $\mathcal{C}$. We also say that $\mathcal{C}$ is an *expansion* of A.
(2) We add "*proper*" to mean that $\mathcal{C}$ is not a reduct of $\mathcal{A}$.
(3) The structures $\mathcal{A}, \mathcal{C}$ are *inter-definable* iff $\mathcal{A}$ is a reduct of $\mathcal{C}$ and vice versa.
(4) A structure $\mathcal{B}$ is a (proper) intermediate structure between $\mathcal{A}$ and $\mathcal{C}$ if $\mathcal{A}$ is a (proper) reduct of $\mathcal{B}$ and $\mathcal{B}$ is a (proper) reduct of $\mathcal{C}$.
(5) $\mathcal{C}$ is a *minimal* expansion of $\mathcal{A}$ if there are no intermediate proper structures.

Motivation

The motivating result for this project is the following theorem of Conant:

Fact 2. [Con18] *(Conant)* $\mathcal{Z}_{\leq} := (\mathbb{Z}, +, 0, 1, \leq)$ *is a minimal expansion of* $\mathcal{Z} := (\mathbb{Z}, +, 0, 1)$.

The proof in [Con18] is self-contained but prior to that, Conant and Pillay [CP18] proved that there are no intermediate stable structures between $\mathcal{Z}$ and $\mathcal{Z}_{\leq}$.

Here is the definition:

Definition 3. A structure M is *unstable* if there is a formula $\varphi(x, y)$ such that for every $n < \omega$ there are $\langle a_i, b_i \mid i < n \rangle$ with $\varphi(a_i, b_j)$ iff $i < j$. Otherwise, M is *stable*.

(Stability is, to put it mildly, one of the most fundamental notions in model theory today.)

Example 4. There are many examples of stable structures including algebraically closed fields, planar graphs, and all abelian groups (in particular $\mathcal{Z}$).

However, clearly $\mathcal{Z}_{\leq}$ is unstable.

Based on Conant and Pillay's result, Alouf and d'Elbée [Ad19] gave another proof of Fact 2, showing that

- There are no proper intermediate unstable structures.

(Their proof was much simpler than Conant's, using the instability.)

However, in their paper Alouf and d'Elbée showed that the same is not true for elementary extensions:

Proposition 5. *If $\mathcal{Z}_{\leq} \prec \mathcal{N}_{\leq}$, and $\gamma \in N$ is a non-standard element then the structure $\mathcal{N}' = (N, +, 0, 1, [0, \gamma])$ is a proper intermediate structure between $\mathcal{N}$ and $\mathcal{N}_{\leq}$. Moreover it is unstable.*

Proof. Being unstable, witnessed by $x - y \in [0, \gamma]$, $\mathcal{N}'$ is a proper expansion of $\mathcal{N}$.

On the other hand, if one can define $\leq$ in $\mathcal{N}'$, then in $\mathcal{N}_{\leq}$ a sentence of the form $\exists\gamma\forall y\,(\theta\,(x, y, \gamma) \leftrightarrow x \leq y)$ holds, where θ can use only the predicate for $[0, \gamma]$ and not γ itself. By elementarity, there is some $\gamma' \in \mathbb{Z}$ satisfying the same for $\mathcal{Z}_{\leq}$. But $[0, \gamma']$ is finite. □

Before continuing, I would like to mention another example:

Example 6. For a prime p, let $\leq_p$ be the p-adic order on $\mathbb{Z}$: $a \leq_p b$ iff the p-adic valuation of a is $\leq$ the p-adic valuation of b, iff any power of p dividing a divides b.

Fact 7. [Ad19] *(Alouf and d'Elbée) The structure $\mathcal{Z}_p := (\mathbb{Z}, +, 0, 1, \leq_p)$ is a minimal expansion of $\mathcal{Z}$.*

Proposition 8. *Let $\mathcal{Z}_p \prec \mathcal{N}_p$ and let $a \in N$ have non-standard p-adic valuation. Let $B = \{b \mid a \leq_p b\}$. Then $(N, +, 0, 1, B)$ is a proper stable intermediate structure.*

It is thus natural to ask:

Question 9. *Suppose that $\mathcal{Z}_{\leq} \prec \mathcal{N}_{\leq}$. Is there a proper intermediate stable structure between $\mathcal{N}$ and $\mathcal{N}_{\leq}$?*

Our results

Theorem 10. *(Alouf, Fornasiero, K.) No: There are no such proper intermediate structures.*

This puts a clear distinction between $\mathcal{Z}_p$ and $\mathcal{Z}_\leq$.

(but there are others: for example $\mathcal{Z}_\leq$ does not eliminate $\exists^\infty$ while $\mathcal{Z}_p$ does, and in fact $\mathcal{Z}_\leq$ is unique in this sense among dp-minimal expansions of $\mathcal{Z}$ by work of Alouf [Alo20].)

Towards the proof, let $\mathcal{A} = \mathcal{N}$,$\mathcal{C} = \mathcal{N}_\leq$ and $\mathcal{B}$ for an intermediate structure.

The first step in the proof is the following observation:

Proposition 11. *The following are equivalent:*

(1) *$\mathcal{B}$ is unstable.*
(2) *$\mathcal{B}$ adds a new unary set: there is some $X \subseteq B$ which is not definable in A.*

Proof. (Sketch) (2) implies (1): Show that one can define in $\mathcal{B}$ an infinite interval $[0, a)$ for some $a \in B \cup \{\infty\}$.

(1) implies (2): By instability, there is some infinite interval $[0, a)$ defined in some elementary extension $\mathcal{B}^*$ of $\mathcal{B}$. From this one can show that an infinite interval is already defined in $\mathcal{B}$. □

In light of this, a restatement of the theorem would be:

Theorem 12. *Suppose that $\mathcal{B}$ defines no new unary sets. Then $\mathcal{B}$ defines no new definable sets.*

We may assume that $\mathcal{C}$ is ω-saturated and fix some such $\mathcal{B}$.

From quantifier elimination we get:

Proposition 13. *$\mathcal{A}$, $\mathcal{C}$ have the same algebraic closure operator:* $\operatorname{acl}_{\mathcal{A}} = \operatorname{acl}_{\mathcal{C}}$. *It follows that* $\operatorname{acl}_{\mathcal{A}} = \operatorname{acl}_{\mathcal{B}}$.

The structure $\mathcal{Z}$ is not just stable, it (really its theory) is in fact weakly minimal, which means superstable of U-rank 1. A well known equivalent definition is:

Definition 14. A complete theory T is *weakly minimal* if, working in a monster model $\mathfrak{C}$, if M is a model and $X \subseteq \mathfrak{C}$ is infinite and definable, then $X \cap M \neq \emptyset$.

Corollary 15. $\operatorname{Th}(\mathcal{B})$ *is weakly-minimal.*

Since every element of $\mathbb{Z}$ is definable, we can conclude

Corollary 16. *(Lascar-Pillay) $\mathcal{B}$ has weak elimination of imaginaries: for any imaginary e there is a real a such that $e \in \operatorname{dcl}^{\mathrm{eq}}(a)$ and $a \in \operatorname{acl}^{\mathrm{eq}}(e)$.*

Definition 17. A stable theory T is *1-based* if for any $a, b \in \mathfrak{C}^{\mathrm{eq}}$, $a \mathop{\smile\!\!\!\!\!\!\;|}_{\operatorname{acl}^{\mathrm{eq}}(a)\cap\operatorname{acl}^{\mathrm{eq}}(b)} b$ where $\mathop{\smile\!\!\!\!\!\!\;|}$ is non-forking independence.

Remark 18. If T has weak elimination of imaginaries then T is 1-based iff for any real tuples $a, b \in \mathfrak{C}$, $a \mathop{\smile\!\!\!\!\!\!\;|}_{\operatorname{acl}(a)\cap\operatorname{acl}(b)} b$.

Fact 19. $\operatorname{Th}(\mathcal{Z})$ *is 1-based.*

Remark 20. In weakly minimal theories, acl has exchange and non-forking independence $\mathop{\smile\!\!\!\!\!\!\;|}$ is the same as algebraic independence.

Since the acl-operator is the same in $\mathcal{A}$ and $\mathcal{B}$, it follows that:

Corollary 21. $\operatorname{Th}(\mathcal{B})$ *is 1-based.*

We will now use a result of Loveys:

Fact 22. [Lov90] *(Loveys) Suppose that $\mathcal{A}$= $(A, +, \ldots)$ is an abelian weakly minimal group of unbounded exponent whose theory is 1-based. Let R be the ring of all definable endomorphisms. Let $\mathcal{B} = (A, +, f)_{f\in R}$. Then $\mathcal{A}$ and $\mathcal{B}$ are inter-definable.*

This result of Loveys is an improvement of a theorem of Hrushovski and Pillay [HP87], which states, without the assumption of weak minimality and unbounded exponent that $\mathcal{A}$ is inter-definable with the structure we get by naming every subgroup of powers of A.

Thus, to finish we must show that every homomorphism definable in $\mathcal{B}$ is definable in $\mathcal{A}$.

Indeed, we finish by the following theorem:

Theorem 23. *(Alouf, Fornasiero, K.) Suppose that $\mathcal{M}$ is a reduct of an ω-saturated structure $\mathcal{N}$, both expand a torsion-free (*) abelian group (**) $(G,+)$. Then if $\mathrm{dcl}_{\mathcal{N}} \subseteq \mathrm{acl}_{\mathcal{M}}$ then every homomorphism $f : G \to G$ definable in $\mathcal{N}$ is definable in $\mathcal{M}$.*

Note that this theorem does not assume anything on the theory.

Remark 24. (*) In fact we do not need the group to be torsion-free. It is enough that it has *small quotients:* For all $n < \omega$, nG has finite index in G.

(**) If the group is not abelian, the theorem holds in the case when the group is an R-group: G is an *R-group* if for all $n < \omega$ and $x, y \in G$, if $x^n = y^n$ then $x = y$.

Final thoughts

Our main theorem can be stated as follows:

Main Theorem 25. *Suppose that L is some language extending the language of abelian groups $\{+\}$. Suppose that $\mathcal{A}$ is an L-structure such that $A \restriction \{+\}$ is an abelian group with small quotients. Furthermore assume that $\mathcal{A}$ is one-based and weakly minimal. Suppose that $\mathcal{B}$ is an expansion to some language L' such that:*

1. *$\mathcal{B}$ is $|L'|^+$-saturated.*
2. *$\mathcal{B}$ does not add new unary subsets to A: if $X \subseteq B$ is definable in B then X is definable in A.*
3. *$\mathrm{acl}_{\mathcal{A}} = \mathrm{acl}_{\mathcal{B}}$.*

Then A and B are inter-definable.

Remark 26. In the case of $\mathcal{B}$ being an intermediate structure between $\mathcal{N}$ and $\mathcal{N}_{\leq}$ we do not need to assume saturation by Proposition 11. The same is true if we replace $\mathcal{Z}_{\leq}$ by $(\mathbb{Q}, +, 0, 1, \leq)$.

Example 27. The assumption that $\mathcal{A}$ has small quotients is needed. Let $F = \mathbb{F}_2$ be the field with two elements, and let $K = F(\alpha)$ where $\alpha^2 + \alpha + 1 = 0$. Let V be an infinite dimensional K-vector space. Let $\mathcal{B} = (V, +, \alpha, c)$ for some $c \neq 0$ and let $\mathcal{A} = (V, +, U, c)$ where $U = \{(x, y) \mid y \in \mathrm{span}_F(\alpha x, c, \alpha c)\}$. Then it is not hard to see that $\mathrm{acl}_{\mathcal{A}} = \mathrm{acl}_{\mathcal{B}}$, both are strongly minimal, but the homomorphism α is not definable in $\mathcal{A}$.

Example 28. The saturation assumption is also needed: Let $\mathcal{B} = (\mathbb{Q}(\pi), f_\pi, +)$ and $\mathcal{A} = (\mathbb{Q}(\pi), +)$ with constants for all elements in $\mathbb{Q}(\pi)$ and $f_\pi : \mathbb{Q}(\pi) \to \mathbb{Q}(\pi)$ is multiplication by π. Then both are strongly minimal with the same (trivial) acl, while f_π is not definable in $\mathcal{A}$.

One can ask what happens in the general strongly minimal case.

Fact 29. [Hru92] *(Hrushovski) Let $\mathcal{F}$ be a strongly minimal expansion of an algebraically closed field $\mathcal{K}$ such that $\mathrm{acl}_{\mathcal{F}} = \mathrm{acl}_{\mathcal{K}}$. Then $\mathcal{F}$ and $\mathcal{K}$ are inter-definable.*

Question 30. *Is there a strongly minimal expansion $\mathcal{K}'$ of an algebraically closed field $\mathcal{K}$ and an expansion $\mathcal{F}$ of $\mathcal{K}'$ such that $\mathrm{acl}_{\mathcal{K}'} = \mathrm{acl}_{\mathcal{F}}$ and $\mathcal{F}$ and $\mathcal{K}'$ are not inter-definable?*

References

[Ad19] Eran Alouf and Christian d'Elbée. A new dp-minimal expansion of the integers. *J. Symb. Log.*, 84(2):632–663, 2019.

[Alo20] Eran Alouf. On dp-minimal expansions of the integers, 2020.

[Con18] Gabriel Conant. There are no intermediate structures between the group of integers and Presburger arithmetic. *J. Symb. Log.*, 83(1):187–207, 2018.

[CP18] Gabriel Conant and Anand Pillay. Stable groups and expansions of $(\mathbb{Z}, +, 0)$. *Fund. Math.*, 242(3):267–279, 2018.

[HP87] U. Hrushovski and A. Pillay. Weakly normal groups. In *Logic colloquium '85 (Orsay, 1985)*, volume 122 of *Stud. Logic Found. Math.*, pages 233–244. North-Holland, Amsterdam, 1987.

[Hru92] Ehud Hrushovski. Strongly minimal expansions of algebraically closed fields. *Israel J. Math.*, 79(2-3):129–151, 1992.

[Lov90] James Loveys. Weakly minimal groups of unbounded exponent. *J. Symbolic Logic*, 55(3):928–937, 1990.