Oka-1 Manifolds

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The motivation – Oka manifolds

A complex manifold X is said to be an **Oka manifold** if it admits many holomorphic maps $S \to X$ from any Stein manifold (or reduced Stein space) S, in the sense that the Oka–Weil approximation theorem and the Oka–Cartan extension theorem hold in the absence of topological obstructions. This class of manifolds developed from the classical **Oka-Grauert-Gromov theory**. It is studied intensively, and its two main characterizations are the following.

- A complex manifold X is an Oka manifold iff every holomorphic map $K \to X$ from a neighbourhood of a compact convex set K in a Euclidean space $\mathbb{C}^n$ is a uniform limit of entire maps $\mathbb{C}^n \to X$ **(F., 2006)**.
- This holds iff every such map $K \to X$ is the core of a dominating holomorphic spray $K \times \mathbb{C}^N \to X$ for some $N \in \mathbb{N}$ **(Kusakabe, 2021)**.

The study of holomorphic curves is a perennial subject in complex geometry. In this work, we introduce and investigate a new class of complex manifolds,

Oka-1 manifolds,

enjoying similar properties for maps from open Riemann surfaces.

A. Alarcón and F. Forstnerič: Oka-1 manifolds. Preprint, March 2023. `http://arxiv.org/abs/2303.15855`.

What is an Oka-1 manifold?

Definition

A connected complex manifold X is an **Oka-1 manifold** if for any

- open Riemann surface R,
- Runge compact set K in R,
- discrete sequence $a_i \in R$ without repetitions,
- continuous map $f : R \to X$ which is holomorphic on a neighbourhood of $K \cup \bigcup_i \{a_i\}$,
- number $\epsilon > 0$, and
- integers $k_i \in \mathbb{N} = \{1, 2, \ldots\}$

there is a holomorphic map $F : R \to X$ which is homotopic to f and satisfies

1. $\sup_{p \in K} \mathrm{dist}_X(F(p), f(p)) < \epsilon$, and
2. F agrees with f to order k_i at the point a_i for every i.

If only (1) holds then X satisfies the Oka-1 property with approximation.

Observations

Every Oka manifold is also an Oka-1 manifold. Let X be an Oka-1 manifold.

- For every point $x \in X$ and tangent vector $v \in T_x X$ there exists an entire map $f : \mathbb{C} \to X$ with $f(0) = x$ and $f'(0) = v$.
- Hence, the Kobayashi pseudometric of X vanishes identically, and every bounded plurisubharmonic function on X is constant, i.e., X is Liouville.

  Moreover, its universal covering manifold is also Liouville.
- Assuming that X is connected, it admits holomorphic maps with everywhere dense images from any open Riemann surface, in particular, from $\mathbb{C}$.
- Conjecturally, no compact complex manifold X of general Kodaira type is Oka-1, since it is believed that any holomorphic line $\mathbb{C} \to X$ in such a manifold is contained in a proper complex subvariety of X (Lang 1986).
- The class of compact Kähler (in particular, compact projective) Oka-1 manifolds is conjecturally related to **Campana special manifolds**.

  See Campana and Winkelmann, *Dense entire curves in rationally connected manifolds, 2019*, `https://arxiv.org/abs/1905.01104`.

Dominability by a family of manifolds

Definition

Let X be a complex manifold, and let $\mathcal{A} = \{W_j\}_{j \in J}$ be a collection of complex manifolds of dimensions $\geq n = \dim X$.

(a) X is **dominable by $\mathcal{A}$ at a point $x \in X$** if there exist $W \in \mathcal{A}$ and a holomorphic map $F : W \to X$ such that $x \in F(W)$ and the differential dF_z at some point $z \in F^{-1}(x)$ is surjective.

(b) X is **densely dominable by $\mathcal{A}$** if there is a closed subset E of X with $\mathscr{H}^{2n-1}(E) = 0$ such that X is dominable by $\mathcal{A}$ at every point $x \in X \setminus E$. (Here, $\mathscr{H}^k$ stands for the k-dimensional Hausdorff measure with respect to a Riemannian metric on X.)

(c) X is **strongly dominable by $\mathcal{A}$** if it is dominable by $\mathcal{A}$ at every point $x \in X$.

Trees and tubes of affine complex lines

An **affine complex line** in $\mathbb{C}^n$ is a set of the form

$$\Lambda = \{a + tv : t \in \mathbb{C}\} = a + \mathbb{C}v$$

where $a \in \mathbb{C}^n$ and $v \in \mathbb{C}^n \setminus \{0\}$ is a **direction vector** of Λ.

A **tree of lines** in $\mathbb{C}^n$ is a connected set $\Lambda = \bigcup_{i=1}^k \Lambda_i$ whose **branches** Λ_i are affine complex lines with linearly independent direction vectors $v_i \in \mathbb{C}^n$. Hence, the **length** k of Λ satisfies $k \leq n$. The tree Λ is a **spanning tree** if $k = n$; equivalently, if the vectors $v_1, \ldots, v_n$ are a basis of $\mathbb{C}^n$.

A **tube of lines** around a tree of lines Λ is an open connected neighbourhood $T \subset \mathbb{C}^n$ of Λ which is a union of affine translates of Λ.

The tube T is **spanning** if the tree Λ is spanning.

A complex manifold X of dimension n is said to be **dominable by tubes of lines (at a point $x \in X$, densely, or strongly)** if X is dominable (at the point $x \in X$, densely, or strongly) by the collection of all spanning tubes of lines in all complex Euclidean spaces of dimension at least n.

The main result

Theorem

A complex manifold which is densely dominable by tubes of lines is Oka-1.

In particular, a complex n-manifold X which is densely dominable by $\mathbb{C}^n$ is Oka-1.

It follows that for a Riemann surface the following conditions are equivalent:

$$\text{Oka-1} \iff \text{non-hyperbolic} \iff \text{Oka.}$$

Note that dense (and even strong) dominability by tubes of lines is an ostensibly weaker condition on a complex manifold than any of the known sufficient conditions in the theory of Oka manifolds.

Problem

(a) *If a complex manifold is strongly dominable by $\mathbb{C}^n$, is it an Oka manifold?*

(b) *Does a tube of lines in $\mathbb{C}^n$ contain a nondegenerate holomorphic image of $\mathbb{C}^n$?*

Applications

Buzzard and Lu (Inventiones Math. 2000) studied dominability of complex surfaces by $\mathbb{C}^2$.

Lárusson and F. (IMRN 2014) studied the question which compact complex surfaces are Oka. The main new classes of Oka-1 manifolds, which are not included in their list but are obtained from our main theorem and the work of Buzzard and Lu, are the following:

Theorem

(a) *Every compact complex surface bimeromorphic to a Kummer surface is densely dominable by $\mathbb{C}^2$, and hence an Oka-1 manifold.*

(b) *Every elliptic K3 surface is an Oka-1 manifold.*

An inspection of the proofs by Buzzard and Lu shows that every such surface is dominable by $\mathbb{C}^2$ at every point in the complement of a divisor (hence, densely dominable), thus an Oka-1 manifold by our main theorem.

I expect that much more can be done in this direction for manifolds of dimension > 2.

Compact rationally connected manifolds are Oka-1

By a different method, we also prove the following result.

Theorem

Every compact rationally connected manifold is an Oka-1 manifold.

In the proof, we use a seminal Runge approximation theorem for maps from compact Riemann surfaces to certain compact (almost) complex manifolds X containing many semipositive rational curves, due to **Gournay (GAFA 2012)**. We also add jet interpolation at finitely many points to Gournay's theorem. This suffices to show the above theorem.

For maps $\mathbb{C} \to X$, a somewhat less precise result was proved by **Campana and Winkelmann**, *Dense entire curves in rationally connected manifolds, 2019*, `https://arxiv.org/abs/1905.01104`.

They used the **comb smoothing theorem** of Kollar, Miyaoka, and Mori (1992).

Complements of thin subsets in Oka-1 manifolds

If X is a complex n-dimensional manifold and E is a closed subset of X with $\mathscr{H}^{2n-2}(E) = 0$, then a generic holomorphic map $M \to X$ from a compact bordered Riemann surface avoids E. This implies

Corollary

Let X be an Oka-1 manifold of dimension n. If E is a closed subset of X with $\mathscr{H}^{2n-2}(E) = 0$, then $X \setminus E$ is Oka-1. This holds in particular if E is a closed complex subvariety of codimension at least two in X.

The hypothesis $\mathscr{H}^{2n-2}(E) = 0$ is optimal. Indeed, the corollary fails in general if E is a complex hypersurface. For example, the complement in $\mathbb{CP}^n$ of the union of $2n+1$ hyperplanes in general position is Kobayashi hyperbolic by Green's theorem.

There is no analogue of this corollary for Oka manifolds, where even the question of removability of a point is open, and discrete sets in $\mathbb{C}^n$ for $n > 1$ are not removable in general.

Theorem "up-down" for Oka-1 manifolds

A holomorphic map $h : X \to Y$ of connected complex manifolds is said to be an **Oka map (Lárusson 2004)** if it enjoys the **parameteric Oka property for liftings** of holomorphic maps from Stein manifolds and is a **Serre fibration**.

$$\begin{tikzcd} & X \arrow[d, "h"] \\ S \arrow[ur, dashed, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

In such case, X is an Oka manifold iff Y is an Oka manifold.

Theorem

Let $h : X \to Y$ be an Oka map between connected complex manifolds.

(a) *If Y is an Oka-1 manifold then X is an Oka-1 manifold.*

(b) *If X is an Oka-1 manifold and the homomorphism $h_* : \pi_1(X) \to \pi_1(Y)$ of fundamentals groups is surjective, then Y is an Oka-1 manifold.*

(c) *If $h : X \to Y$ is a holomorphic fibre bundle with a connected Oka fibre, then X is an Oka-1 manifold if and only if Y is an Oka-1 manifold.*

Disc approximation property implies Oka-1 properties

We now explain the proof of the Main Theorem.

A holomorphic map $R \to X$ is constructed as a limit $f = \lim_{j\to\infty} f_j : K_j \to X$ with respect to an exhaustion of the open Riemann surface R by an increasing sequence of smoothly bounded Runge domains K_j. The change of topology can easily be handled by Mergelyan theorem.

The main problem is to "fatten" the domain of a holomorphic map, which amounts to approximately extending a given map across an attached disc.

Proposition

Let X be a connected complex manifold.

(a) *Assume that for any open Riemann surface R and pair of compact sets $K \subset L$ in R with piecewise smooth boundaries such that $D = L \setminus \mathring{K}$ is a disc attached to K along an arc $\alpha \subsetneq bD$, every holomorphic map $f : K \to X$ can be approximated uniformly on K by holomorphic maps $\tilde{f} : L \to X$. Then, X has the Oka-1 property with approximation.*

(b) *If in addition the map $\tilde{f}$ can be chosen to agree with f to a given order at a given finite set of points in $\mathring{K}$, then X is an Oka-1 manifold.*

Extending across a bump assuming values in a tube

The first step in the proof of the Main Theorem is the following lemma, which explains why a spanning tube of lines in $\mathbb{C}^n$ is an Oka-1 manifold.

Lemma

Assume that K and $L = K \cup D$ are as in the Proposition, with $D = L \setminus \mathring{K}$ a disc attached to K along an arc $\alpha \subsetneq bD$.

Let $f = (f_1, \ldots, f_n) : K \to \mathbb{C}^n$ be a holomorphic map, and let $T \subset \mathbb{C}^n$ be a spanning tube of lines such that $f(\alpha) \subset T$.

Then we can approximate f as closely as desired uniformly on K and interpolate it to any given finite order at any given finite set of points in $\mathring{K}$ by holomorphic maps $\tilde{f} : K \cup D \to \mathbb{C}^n$ such that $\tilde{f}(D) \subset T$.

Proof, 1

Let $\Delta^n \subset \mathbb{C}^n$ denotes the unit polydisc. We shall first prove the lemma under the following additional assumptions on f and T:

(a) $f(\alpha) \subset r\Delta^n$ for some $r > 0$, and

(b) Λ is a tree of lines in the normal form

$$\Lambda = \mathbb{C}e_n \cup \bigcup_{j=1}^{l} \Lambda^j,$$

where each Λ^j is a tree in coordinate directions such that $\Lambda^j \cap \mathbb{C}e_n = a_j e_n$ for some numbers $a_1, \ldots, a_l \in \mathbb{C}$.

(c) $T \subset \mathbb{C}^n$ is the polydisc tube of radius r around Λ.

These conditions on f and T imply that

$$f(\alpha) \subset r\Delta^n \subset T.$$

Proof, 2

We first consider the case when Λ is a simple tree (a comb), so $l = n - 1$ and every Λ^j is a branch.

We begin by explaining how to choose the first $n-1$ components of the new map

$$\tilde{f} = (\tilde{f}', \tilde{f}_n) = (\tilde{f}_1, \dots, \tilde{f}_n) : K \cup D \to \mathbb{C}^n.$$

The last component $\tilde{f}_n$ will be determined in the final step.

Proof, 3

Let $\beta = bD \setminus \alpha$ be the complementary arc to α in bD. Pick a closed disc $\Delta_0 \subset D$ such that $\Delta_0 \cap \alpha = \varnothing$ and $\Delta_0 \cap bD$ is an arc contained in β.

We extend the first component f_1 of f to $K \cup \Delta_0$ by setting $f_1 = 0$ on Δ_0.

By Runge theorem we can approximate f_1 on $K \cup \Delta_0$ by a holomorphic function $\tilde{f}_1$ on $L = K \cup D$ such that $|\tilde{f}_1| < r$ holds on $\alpha \cup \Delta_0$.

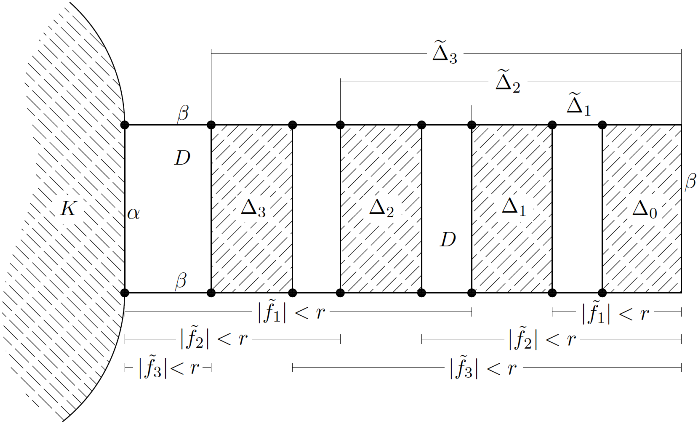
Hence, there is a closed disc $\Delta_1 \subset D$ such that

- (i_1) $\Delta_1 \cap (\alpha \cup \Delta_0) = \varnothing$,
- (ii_1) $\overline{D \setminus \Delta_1}$ is the union of two disjoint discs, and
- (iii_1) $|\tilde{f}_1| < r$ holds on $\overline{D \setminus \Delta_1}$.

Condition (iii_1) holds if the disc Δ_1 satisfying conditions (i_1) and (ii_1) is chosen large enough.

Note that $K \cap \Delta_1 = \varnothing$, and hence $K \cup \Delta_1$ is Runge in $L = K \cup D$.

Illustration



Proof, 4

If $n = 2$, we proceed to the final argument explaining how to choose $\tilde{f}_n$.

Assume now that $n > 2$. Let $\widetilde{\Delta}_1$ denote the union of Δ_1 and the component of $D \setminus \Delta_1$ containing Δ_0.

We extend the second component f_2 of f to $K \cup \widetilde{\Delta}_1$ by taking $f_2 = 0$ on $\widetilde{\Delta}_1$. Next, we apply Runge theorem on $K \cup \widetilde{\Delta}_1$ to find a holomorphic function $\tilde{f}_2$ on L such that $|\tilde{f}_2| < r$ holds on $\alpha \cup \widetilde{\Delta}_1$.

Hence, there is a disc $\Delta_2 \subset D$ such that

- i_2 $\Delta_2 \cap (\alpha \cup \widetilde{\Delta}_1) = \varnothing$,
- ii_2 $\overline{D \setminus \Delta_2}$ is the union of two disjoint discs, and
- iii_2 $|\tilde{f}_2| < r$ holds on $\overline{D \setminus \Delta_2}$.

Proof, 5

Continuing inductively, we approximate the first $n-1$ components of f by holomorphic functions $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$ on L such that

$$|\tilde{f}_i| < r \ \text{ holds on } \ \overline{D \setminus \Delta_i} \ \text{ for } i = 1, \ldots, n-1, \tag{1}$$

where $\Delta_1, \ldots, \Delta_{n-1}$ are pairwise disjoint closed discs in $D \setminus (\alpha \cup \Delta_0)$ as shown in the illustration.

We now extend the last component f_n to the Runge compact set $K' = K \cup \bigcup_{i=0}^{n-1} \Delta_i$ by setting

$$f_n = a_i \text{ on } \Delta_i \text{ for } i = 0, 1, \ldots, n-1,$$

where $a_0 = 0$ and the numbers $a_i \in \mathbb{C}$ for $i = 1, \ldots n-1$ are the z_n-coordinates of the branches Λ^i of the tree T.

By Runge theorem, we approximate f_n on K' by a holomorphic function $\tilde{f}_n$ on $L = K \cup D$ such that

$$|\tilde{f}_n - a_i| < r \ \text{ holds on } \ \Delta_i \ \text{ for } i = 0, 1, \ldots, n-1. \tag{2}$$

Proof, 6

Summary:

Conditions (1) and (2) imply that the holomorphic map

$$\tilde{f} = (\tilde{f}', \tilde{f}_n) = (\tilde{f}_1, \ldots, \tilde{f}_n) : K \cup D \to \mathbb{C}^n$$

sends the disc D into the tube T.

Indeed, on the disc Δ_i for $i = 1, \ldots, n-1$ all components of $\tilde{f}'$ except $\tilde{f}_i$ are smaller than r in absolute value while $|\tilde{f}_n - a_i| < r$, so $\tilde{f}(\Delta_i)$ is contained in the polydisc tube of radius r around the affine line $a_n e_n + \Lambda_i \subset \Lambda$.

On the other hand, on $D \setminus \bigcup_{i=1}^{n-1} \Delta_i$ all components of $\tilde{f}'$ are smaller than r, so its image by $\tilde{f}$ is contained in the polydisc tube of radius r around the stem $\Lambda_n = \mathbb{C} e_n \subset \Lambda$. Note also that

$$|\tilde{f}_j| < r \text{ holds on } \alpha \cup \Delta_0 \text{ for all } j = 1, \ldots, n. \tag{3}$$

The case when T is not a simple tree is handled by induction, applying the above procedure to every subtree T^j of T.

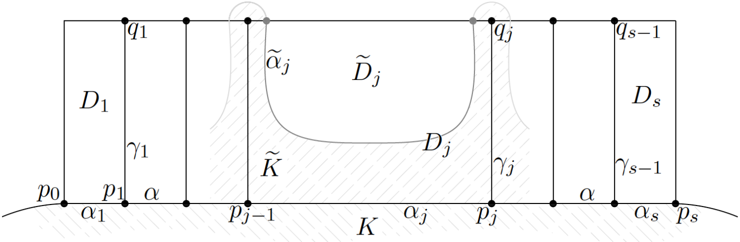
Proof, 7

It remains to prove the general case of the lemma with the only assumption that $f(\alpha) \subset T$, where $T \subset \mathbb{C}^n$ is a spanning tube of lines.

We subdivide the arc α to finitely many subarcs so that on each of them we have the situation of the special case.

We also subdivide the disc D into subdiscs D_j by the arcs γ_j as in the figure, extend f as constant on each γ_j, and approximate it by a holomorphic map on a neighbourhood of $K \cup \bigcup_j \gamma_j$.

We can then approximately extend the new map to every subdisc $\widetilde{D}_j \subset D_j$ without affecting what is done on adjacent discs. This localizes the problem.



Recall the main theorem

Theorem

A complex manifold X which is densely dominable by tubes of lines is an Oka-1 manifold.

So far, we have proved that a spanning tube of lines $T \subset \mathbb{C}^n$ is Oka-1.

The proof of the general case uses this special case together with gluing techniques used in Oka theory.

Proof, 1

Let $f : K \to X$ be a holomorphic map from a compact domain $K \subset R$, and let $L \supset K$ be such that $\overline{L \setminus K}$ is a union of annuli.

After a small perturbation of f, we may assume that $f(bK)$ is contained in the domain of X which is dominable by tubes of lines.

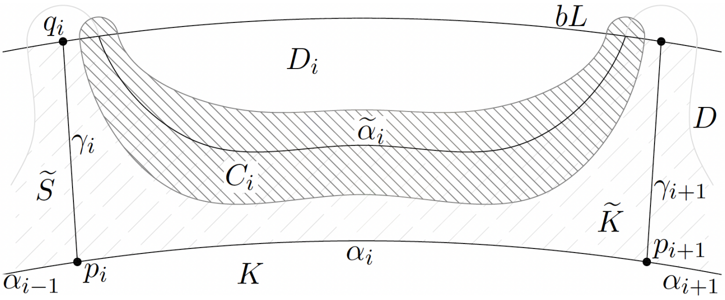
Hence, we can split bK into a union of compact subarcs $\{\alpha_i : i \in \mathbb{Z}_l\} = \mathbb{Z}/l\mathbb{Z} = \{0, 1, \ldots, l-1\}$ such that

- A1 α_i and α_{i+1} have a common endpoint p_{i+1} and are otherwise disjoint for every $i \in \mathbb{Z}_l$,
- A2 $\bigcup_{i \in \mathbb{Z}_l} \alpha_i = bK$, and
- A3 for every $i \in \mathbb{Z}_l$ there are a spanning tube of lines $T_i \subset \mathbb{C}^{n_i}$ for some $n_i \geq \dim X$, a holomorphic map $\sigma_i : T_i \to X$, a neighbourhood $U_i \subset X$ of $f(\alpha_i)$, and an open subset $\omega_i \subset T_i$ such that $\sigma_i(\omega_i) = U_i$ and the triple $(\omega_i, \sigma_i, U_i)$ is a submersion chart.

Proof, 2

Let p_i and p_{i+1} denote the endpoints of α_i, ordered so that $p_{i+1} = \alpha_i \cap \alpha_{i+1}$ for each $i \in \mathbb{Z}_l$. Choose an embedded arc $\gamma_i \subset D$ connecting the point p_i to a point $q_i \in bL$ so that these arcs are pairwise disjoint, they intersect $bD = bK \cup bL$ only at the respective endpoints p_i and q_i, and these intersections are transverse. Set

$$S = K \cup \bigcup_{i \in \mathbb{Z}_l} \gamma_i$$



Proof, 3

Recall that $f(\alpha_i) \subset U_i$. We extend f as a constant map to each arc γ_i having the value $f(p_i)$.

By Mergelyan theorem, we approximate the resulting map $f : S \to X$ by a holomorphic map $V \to X$ on a neighbourhood $V \subset R$ of S so that

$$f(\gamma_i \cup \alpha_i \cup \gamma_{i+1}) \subset U_i \text{ holds for each } i \in \mathbb{Z}_l.$$

Let $\widetilde{S} \subset V$ be a compact neighbourhood of S with smooth boundary and

$$\widetilde{K} := L \cap \widetilde{S} \subset V.$$

We can choose $\widetilde{S}$ (and hence $\widetilde{K}$) such that the set $\overline{L \setminus \widetilde{K}} = \bigcup_{i \in \mathbb{Z}_l} D_i$ is the union of pairwise disjoint compact discs D_i with piecewise smooth boundaries, and for each $i \in \mathbb{Z}_l$ the arc

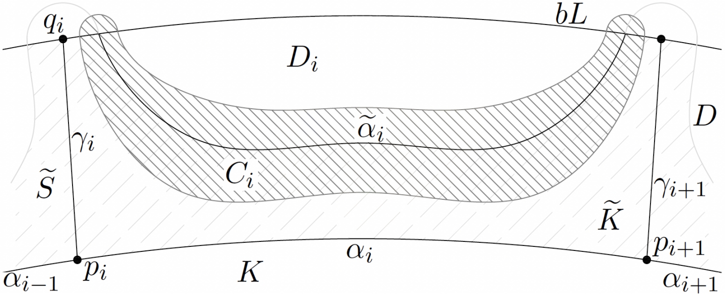
$$\widetilde{\alpha}_i = \overline{bD_i \cap \mathring{L}}$$

is so close to the arc $\gamma_i \cup \alpha_i \cup \gamma_{i+1}$ that

$$f(\widetilde{\alpha}_i) \subset U_i \quad \text{for all } i \in \mathbb{Z}_l.$$

Proof, 3

Choose a neighbourhood C_i of $\widetilde{\alpha}_i$ so that $f(C_i) \subset U_i$.



Recall that there are a spanning tube of lines $T_i \subset \mathbb{C}^{n_i}$ and a holomorphic map $\sigma_i : T_i \to X$ such that $\sigma_i(\omega_i) = U_i$ and $(\omega_i, \sigma_i, U_i)$ is a submersion chart.

Hence, we can lift f over C_i to a map $F_i : C_i \to T_i$ so that $\sigma_i \circ F_i = f_i$ on C_i. We apply the special case to approximate F_i by a map $\widetilde{F}_i : C_i \cup D_i \to T_i$, and then glue $\sigma_i \circ \widetilde{F}_i$ with f. (To be precise, we need to work with dominating sprays of maps.) Doing this finitely many times extends f from K to L.

This completes the proof of the Main Theorem.

$\sim$ **Thank you for your attention** $\sim$