# Pluripotential theory: a synthetic approach

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# A (biased) review of potential theory

Consider a compact Riemann surface $X$ with a Kähler form $\omega$ of mass $1$.

- Each probability measure on $X$ is of the form $\mu = \omega_\varphi := \omega + \mathrm{dd^c}\,\varphi$ where $\varphi$ $\omega$-**subharmonic** function.
- The **energy** $\mathrm{J}(\mu) := \frac{1}{2}\int(-\varphi)\,\mathrm{dd^c}\,\varphi \in [0, +\infty]$ is finite iff $\nabla\varphi \in L^2$.
- The space $\mathcal{M}^1 := \{\mu \mid \mathrm{J}(\mu) < \infty\}$ of **measures of finite energy** is complete with respect to **quasi-metric** $\delta(\mu, \nu) := \mathrm{J}(\varphi - \psi)$, where $\mu = \omega_\varphi$, $\nu = \omega_\psi$.
- Variational formulation:
$$\mathrm{J}(\mu) = \sup_{\varphi\,\omega\text{-sh}} \left(\mathrm{E}(\varphi) - \int \varphi\,\mu\right)$$
where $\mathrm{E}(\varphi) := \frac{1}{2}\int \varphi(\omega + \omega_\varphi)$ primitive of $\varphi \mapsto \omega_\varphi$.
- Shows $\mu \mapsto \mathrm{J}(\mu)$ convex lsc on (compact) space of probability measures.
- In fact, strictly convex $\Rightarrow$ existence of **equilibrium measure** $\mu_K$ for any (non-polar) compact $K \subset X$ (unique minimizer of $\mathrm{J}(\mu)$ with $\operatorname{supp}\mu \subset K$).

# Synthetic pluripotential formalism

Consider a compact topological space $X$, equipped with:

- a dense linear subspace $\mathcal{D} \subset \mathrm{C}^0(X)$ of **test functions** $\varphi$, containing the constants;
- a partially ordered vector space $\mathcal{Z}$ of **closed** $(1,1)$-**forms**, with a linear map $\mathrm{dd^c} \colon \mathcal{D} \to \mathcal{Z}$ vanishing on constants;
- a nonzero $n$-linear symmetric map $\mathcal{Z}^n \ni (\theta_1, \dots, \theta_n) \mapsto \theta_1 \wedge \cdots \wedge \theta_n$ to signed Radon measures on $X$, assumed to be positive for $\theta_i \in \mathcal{Z}_+$, and such that bilinear form

  $$\mathcal{D} \times \mathcal{D} \to \mathbb{R} \quad (\varphi, \psi) \mapsto \int \varphi \, \mathrm{dd^c} \, \psi \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1}$$

  is symmetric (**integration-by-parts**), and seminegative when $\theta_i \geq 0$ (**Hodge index** condition).

We then introduce the **Bott–Chern space** $\mathrm{H_{BC}}(X) := \mathcal{Z} / \, \mathrm{dd^c} \, \mathcal{D}$, with its **positive cone** = interior of the image of $\mathcal{Z}_+$.

# Main examples

- **Kähler case:** $X$ = compact Kähler manifold; $\mathcal{D} = C^\infty(X)$, $\mathcal{Z}$ = usual space of closed $(1,1)$-forms. Then $\mathrm{H}_{\mathrm{BC}}(X) = H^{1,1}(X,\mathbb{R})$, positive cone = Kähler cone. For $\varphi \in \mathcal{D}$ and $\theta_i \in \mathcal{Z}_+$ have

$$\int \varphi \, \mathrm{dd^c}\varphi \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} = -\int \mathrm{d}\varphi \wedge \mathrm{d^c}\varphi \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} \leq 0.$$

- **Non-Archimedean case:** $X$ = projective Berkovich space over NA field; $\mathcal{D}$ = PL functions $\leftrightarrow$ divisors on models of $X$ over valuation ring. Then $\mathrm{H}_{\mathrm{BC}}(X) = \mathrm{N}^1(X)$, positive cone = ample cone.

- **Toric case:** $X$ = compactification of $\mathbb{R}^n$ wrt rational fan $\Sigma$, and set:
  - $\mathrm{PL}_\Sigma$ = $\mathbb{Q}$-PL functions on $\mathbb{R}^n$ wrt to rational polyhedral decomposition with recession fan $\Sigma$;
  - $\mathcal{D}$ = bounded functions in $\mathrm{PL}_\Sigma$;
  - $\mathcal{Z} = \mathrm{PL}_\Sigma$ modulo affine functions, with $\mathrm{dd^c}\colon \mathcal{D} \to \mathcal{Z}$ obvious map;
  - $\mathcal{Z}_+$ = convex functions in $\mathrm{PL}_\Sigma$ modulo affine functions;
  - wedge product map induced by mixed real Monge–Ampère operator.

# Monge–Ampère operator and Dirichlet functional

Back to general setting. Set $\mathcal{M}$ = space of probability measures on $X$.

- Fix $\omega \in \mathcal{Z}_+$ such that $[\omega] > 0$ in $\mathrm{H}_{\mathrm{BC}}(X)$, with volume $V := \int_X \omega^n > 0$. .
- Space of $\omega$-**psh test functions** $\mathcal{D}_\omega := \{\varphi \in \mathcal{D} \mid \omega_\varphi := \omega + \mathrm{dd^c}\,\varphi \geq 0\}$.
- Define the **Monge–Ampère operator** $\mathrm{MA}\colon \mathcal{D}_\omega \to \mathcal{M}$ by $\mathrm{MA}(\varphi) := V^{-1}\omega_\varphi^n$. It admits a primitive $\mathrm{E}(\varphi) = \frac{1}{n+1}\sum_{j=0}^n V^{-1}\int \varphi\,\omega_\varphi^j \wedge \omega^{n-j}$.
- Hodge index condition $\Rightarrow$ $\mathrm{E}$ concave on $\mathcal{D}_\omega$ $\Leftrightarrow$ nonnegativity of the **Dirichlet functional**

$$\mathrm{J}(\varphi,\psi) := \mathrm{E}(\varphi) - \mathrm{E}(\psi) + \int (\psi - \varphi)\,\mathrm{MA}(\varphi).$$

- For $n = 1$, $\mathrm{J}(\varphi,\psi) = \frac{1}{2}\int(\varphi-\psi)\,\mathrm{dd^c}(\psi-\varphi)$. In general, $\mathrm{J}(\varphi,\psi)$ positive linear combination of $\int(\varphi-\psi)\,\mathrm{dd^c}(\psi-\varphi)\wedge\omega_\varphi^j\wedge\omega_\psi^{n-j}$, $j = 0,\dots,n$.
- Dirichlet functional is **quasi-symmetric**, i.e. $\mathrm{J}(\varphi,\psi) \approx \mathrm{J}(\psi,\varphi)$, and satisfies **quasi-triangle inequality** $\mathrm{J}(\varphi,\psi) \lesssim \mathrm{J}(\varphi,\tau) + \mathrm{J}(\tau,\psi)$ (BBEGZ).

# Measures of finite energy

- Define the **energy** of $\mu \in \mathcal{M}$ as $\mathrm{J}(\mu) := \sup_{\varphi \in \mathcal{D}_\omega}(\mathrm{E}(\varphi) - \int \varphi\, \mu) \in [0, +\infty]$. Then $\mathrm{J}\colon \mathcal{M} \to [0, +\infty]$ is convex and lsc.
- Endow the space of **measures of finite energy** $\mathcal{M}^1 := \{\mu \in \mathcal{M} \mid \mathrm{J}(\mu) < \infty\}$ with the **strong topology** $=$ coarsest refinement of weak topology such that $\mathrm{J}\colon \mathcal{M}^1 \to \mathbb{R}$ continuous.
- If $\mu = \mathrm{MA}(\varphi)$ with $\varphi \in \mathcal{D}_\omega$, then concavity of $\mathrm{E} \Rightarrow \mathrm{J}(\mu) = \mathrm{J}(\varphi, 0) < \infty$.

**Theorem (BJ23)**

*Assume $\omega$ has the **orthogonality property**. Then:*

(i) $\mathrm{MA}\colon \mathcal{D}_\omega \to \mathcal{M}^1$ *has dense image;*

(ii) *the strong topology of $\mathcal{M}^1$ is defined by a unique quasi-metric $\delta$ such that*

$$\delta(\mathrm{MA}(\varphi), \mathrm{MA}(\psi)) = \mathrm{J}(\varphi, \psi) \quad \text{for} \quad \varphi, \psi \in \mathcal{D}_\omega;$$

(iii) *the quasi-metric space $(\mathcal{M}^1, \delta)$ is complete.*

# Comments

- **Kähler case**: orthogonality property holds; amounts to

$$\int (f - \mathrm{P}(f))\,\mathrm{MA}(\mathrm{P}(f)) = 0$$

  for all $f \in \mathcal{D}$ with $\omega$-**psh envelope** $\mathrm{P}(f)$ (and Theorem known in that case, as a consequence of BBEGZ).
- **NA (and hence toric) case**: orthogonality property also holds (B–Gubler–Martin).
- Density of the image of $\mathrm{MA}$: pick $\mu \in \mathcal{M}^1$, and a **maximizing sequence for** $\mu$, i.e. a sequence $(\varphi_j)$ in $\mathcal{D}_\omega$ that computes $\mathrm{J}(\mu) = \sup_\varphi(\mathrm{E}(\varphi) - \int \varphi\,\mu)$.
- Orthogonality property $\Rightarrow$ uniform differentiability of Legendre transform of energy $\Rightarrow$ $\mathrm{MA}(\varphi_j) \to \mu$.
- Quasi-metric $\delta$ defined on (dense) image of $\mathrm{MA}$. Extended to $\mathcal{M}^1$ by uniform continuity, using Hölder estimates for mixed MA integrals derived from iterated application of Cauchy–Schwarz inequality (Hodge index condition). Strategy going back to Błocki, BBEGZ etc...

# Varying $\omega$

From now on assume orthogonality property.

- $\mathcal{M}^1 = \mathcal{M}^1_\omega$ only depends on $[\omega] \in \mathrm{H}_{\mathrm{BC}}(X)$, but not independent of $[\omega]$ in general (e.g. non-connected Riemann surface).
- Say **submean value property** holds if $\sup \varphi \leq V^{-1} \int \varphi \, \omega^n + C$ for all $\varphi \in \mathcal{D}_\omega$ and a uniform constant $C$. Condition independent of $\omega$, and holds iff $X$ **irreducible** in Kähler and NA cases.

**Theorem (BJ23)**

*Assume the submean value (and orthogonality) property. Then:*

(i) *$\mathcal{M}^1$ is independent of $\omega$ (as a topological space);*

(ii) *for each $\theta \in \mathcal{Z}$, there exists $\mathrm{J}^\theta_\omega \colon \mathcal{M}^1 \to \mathbb{R}$ continuous such that $\mathrm{J}^\theta_\omega(\mu) = \frac{d}{dt}\big|_{t=0} \mathrm{J}_{\omega+t\theta}(\mu)$ for $\mu \in \mathcal{M}^1$ (**twisted energy**);*

(iii) *$\omega \mapsto \mathrm{J}^\theta_\omega(\mu)$ is Hölder continuous.*

# Application to cscK metrics and K-stability

- Assume first $X$ compact Kähler manifold. Smooth metric $\rho$ on canonical bundle $K_X \leftrightarrow$ volume form $\mu_\rho \rightsquigarrow$ **entropy** $\mathrm{Ent}\colon \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$, such that $\mathrm{Ent}(\mu) := \int \log\left(\frac{\mu}{\mu_\rho}\right)\mu$ if $\mu \ll \mu_\rho$ and $\infty$ otherwise.
- Define **free energy** $\mathrm{F}_\omega\colon \mathcal{M}^1 \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathrm{F}_\omega(\mu) = \mathrm{Ent}(\mu) + \mathrm{J}^\theta_\omega(\mu)$$

  with $\theta \in \mathcal{Z}$ curvature of $\rho$.
- Free energy independent of $\rho$ (up to additive constant), and $\mathrm{F}_\omega \circ \mathrm{MA} =$ **Mabuchi K-energy** on $\mathcal{D}_\omega$.
- Chen–Cheng: there exists a unique cscK metric in $[\omega] \iff \mathrm{F}_\omega$ **coercive**: $\mathrm{F}_\omega \geq \varepsilon\, \mathrm{J}_\omega - C$ with $\varepsilon, C > 0$.
- NA case: PL metric $\rho$ on $K_X \rightsquigarrow$ NA entropy $\mathrm{Ent} \rightsquigarrow$ free energy $\mathrm{F}_\omega = \mathrm{Ent} + \mathrm{F}^\theta_\omega$ with $\theta \in \mathcal{Z}$ curvature of $\rho$.
- Coercivity of $\mathrm{F}_\omega \Leftrightarrow$ **(strong) K-stability** of $(X, \omega)$.

# Openness for cscK metrics and K-stability

**Theorem (BJ23)**

*Assume $X$ compact Kähler or projective Berkovich space. Then coercivity of $\mathrm{F}_\omega$ is an open condition wrt $\omega$.*

- Kähler case: openness of unique cscK metrics (Lebrun-Simanca);
- NA case: openness of strong K-stability;
- actually show that **twisted coercivity threshold**

$$\sup\{\sigma \in \mathbb{R} \mid F + \mathrm{J}_\omega^\theta \geq \sigma\, \mathrm{J}_\omega + A \text{ for some } A \in \mathbb{R}\}$$

continuous function of $(\omega, \theta)$ for any $F \colon \mathcal{M}^1 \to \mathbb{R} \cup \{+\infty\}$.

*Thanks for your attention, et joyeux anniversaire László !*