A tour through Laszlo's mathematics.

Bo Berndtsson

La metrique de Kobayashi.(Bull Soc Math France -81)

Let D be a domain in $\mathbb{C}^n$ (or a complex manifold), and let p be a point in D. Let $X \in T_p^{1,0}$ be a tangent vector at p. The Kobayashi norm of X is defined by looking at all maps from the unit disk U to D such that

$$f : U \to D, \quad f(0) = p, \quad f'(0) = \lambda X, \lambda > 0$$

and putting

$$\|X\|^{-1} = \sup_f \lambda.$$

This is perhaps the most basic manifestation of the link between complex analysis and hyperbolic metrics. In general it is not so well behaved but Laszlo had the idea to restrict attention to convex domains D (or 'lineally convex'), where he could prove that the theory is almost 'perfect'.

Extremal disks

Let us say that a disk (i.e. a mapping f as above) is extremal if the sup in the definition is attained. Fix the point p in D and look at all the extremal disks passing through p. Then the extremal maps are injective and

1. The extremal disks form a foliation of $D \setminus \{p\}$

2. They induce a map from D to the unit ball B_n which extends to a homeomorphism between the closures of D and B_n and is smooth outside p if the boundary of D is sufficiently smooth.

3. Put $U(z) = \log |F(z)|$. Then U is a plurisubharmonic Green's function for D; it vanishes on the boundary, is plurisubharmonic and solves

$$MA(U) = 0$$

outside p and has a log-singularity at p.

Consequences

What I have tried to describe is the infinitesimal version of the Kobayashi metric. Laszlo also discusses the general Kobayashi pseudometric, defined by looking at extremal maps passing through two distinct points. This is in general just a pseudometric, but Laszlo proves that in the case of (lineally) convex domains it is a bona fide metric.

As Laszlo argues, the function F (the ‘Riemann-Lempert function’) is probably as close as one can get in several variables to a Riemann mapping function.

The regularity of F gives a proof of Fefferman’s famous theorem of smooth extensions to the boundary of biholomorphisms between strictly pseudoconvex domains - even if the domains are not convex.

Symmetries and other transformations of the complex Monge-Ampere equation. (Duke -85)

Laszlo's work on the Kobayashi metric gave in particular a construction of plurisubharmonic Green's functions for convex domains. Maybe part of the motivation for the second paper I will discuss was the similar problem for Green's functions for the complement of such domains. That would mean a function U defined in D^c with boundary values zero, such that

$$MA(U) = 0, \quad U(z) \sim \log|z| + C, \ |z| \to \infty.$$

This is in a way an even more natural problem than the interior Green's function. We recall briefly the one-variable situation.

Green's functions and capacity in one variable.

Let K be a compact set in $\mathbb{C}$ with connected complement. The equilibrium measure on K is a probability measure supported on K that minimizes the potential (or electric) energy

$$E(\mu) = \int_{K\times K} \log \frac{1}{|z-w|} d\mu(z) d\mu(w).$$

It is a classical fact from potential theory (Gauss, Frostman ...) that the logarithmic potential of such a measure

$$p(\mu)(z) = \int_K \log \frac{1}{|z-w|} d\mu(w),$$

is constant on K. The constant value γ, which coincides with the minimal energy, is the capacity of K.

Continuation

Moreover

$$-p(\mu)(z) \sim \log|z|, \; z \to \infty.$$

Therefore

$$U(z) := \gamma - p(z)$$

is the exterior Green's function and the capacity can be read off from its behaviour at infinity.

In one variable the difference between exterior domains and interior domains is not so great, but in several variables the difference is great. How did Laszlo overcome this problem?

The baby case of Laszlo's symmetry

is still very interesting. Let $\phi(z)$ be a smooth strictly convex function in $\mathbb{C}^n$. We define its Legendre transform by

$$\hat{\phi}(w) = \sup_z 2\mathrm{Re}\, z \cdot \bar{w} - \phi(z).$$

The sup is attained where $\bar\partial$ of the right hand side vanishes, i. e. when

$$w = \partial\phi/\partial\bar{z} =: \lambda(z).$$

Theorem

$$\lambda^*(\partial\bar\partial\hat{\phi}) = \partial\bar\partial\phi.$$

This theorem is very surprising since λ is not holomorphic and $\lambda^*\hat{\phi}$ is not equal to ϕ!

An interpretation of the baby theorem

Let $t \to \phi_t$ be a smooth curve of convex functions, and $\hat{\phi}_t$ their Legendre transforms. Then it's a classical fact that

$$\lambda^*((d/dt)\hat{\phi}_t) = -(d/dt)\phi_t.$$

This together with the theorem implies that

$$\int |(d/dt)\hat{\phi}_t|^2 MA(\hat{\phi}_t) = \int |d/dt)\phi_t|^2 MA(\phi_t).$$

In the time-honoured language of geometry of the Mabuchi space, this says that the Legendre transform is an isometry on (an open subset of) the Mabuchi space.

Hence, geodesics are mapped to geodesics! In the language of Riesz-Thorin interpolation, this amounts to the so called duality principle; the duals of an interpolating family of norms is again interpolating.

the general case

is more difficult! It is based on general canonical transformations between the cotangent bundles of two complex manifolds. One way to understand it is perhaps to consider more general Legendre-type transformations

$$\sup_z 2\mathrm{Re}\, G(z, w) - \phi(z),$$

where G is the generating function of a canonical transformation. In any case, going back to the exterior Monge-Ampere equation, one starts by looking at the 'complex polar' of D, the set of complex hyperplanes

$$w \cdot z = 1$$

that do not intersect D. In one variable, this is the complement, in several it can be mapped to the complement by one of Laszlo's symmetries.

3-dimensional CR-manifolds (JAMS -92, Math Ann -94, Inventiones -95)

These papers concerns the embeddability problem of abstract strictly pseudoconvex 3-dimensional CR-manifolds, i. e. compact 3-dimensional manifolds (without boundary) that could conceivably appear as boundaries of complex surfaces in some $\mathbb{C}^n$. This was (is?) known as avery subtle problem; more delicate than the corresponding case of dimensions 5, 7 etc., as embeddability in 3-dimensions is rare. Two problems are central:

1. Is the set of embeddable CR-structures on a given manifold M closed?

2. Are embeddings stable? In other words, if M_0 is embeddable and M' is a nearby CR-structure on the same manifold which is also embeddable, can it then be embedded close to the embedding of M_0?

In the three papers mentioned above these questions are approached with three different methods.

In the first paper, the embddability problem is related to the existence of certain S^1-actions on the manifold. This condition is too complicated to describe here, but comes surprisingly close to a characterization. It has the following concrete consequence: In the special case when M_0 is the boundary of a lineally convex domain in $\mathbb{C}^2$, any nearby embeddable structure comes from a nearby lineally convex domain in $\mathbb{C}^2$.

In the second paper, this is generalized to general strongly pseudoconvex boundaries in $\mathbb{C}^2$. The methods here are completely different and based on $\bar{\partial}$-techniques, which in fact leads to a stronger approximation result for *CR*-functions.
The methods of the third paper are again completely different. They are based on a general result on algebraic approximation of complex analytic maps between algebraic manifolds. One striking consequence is that if a strongly pseudoconvex CR-manifold can be 'filled' from the inside, it can also be filled from the outside, i.e. it is the pseudococave boundary of another manifold

Grauert tubes and Monge-Ampere equations. (Joint work with Szoke, Math Ann -91, Bull London Math Soc -12

The background of this paper is a result by Stoll who considered Stein manifolds on which there is defined a 'log-like' function u. This means that $u \geq 0$ is proper, e^u is smooth and strictly psh, and

$$MA(u) = 0$$

outside a , say, unique point where u has a log-singularity. Stoll then showed that X must be biholomorphic to $\mathbb{C}^n$ and $u = \log|z|$.

Following Patrizio and Wong, Lempert and Szoke considered the analogous situation when the single point p is replaced by a smooth manifold M. We have a proper psh function $u \geq 0$ such that its minimum set where $u = 0$ is now a smooth compact submanifold of X.

Like in Stoll's case we assume that

$$MA(u) = 0$$

outside of M, but now the regularity assumption is that u^2 is smooth and strictly psh (instead of e). Then $i\partial\bar{\partial}u^2$ defines a Kahler metric on X, that has a restriction, g, to M. Laszlo and Robert prove that g must have non-negative curvature, and that X is determined by (M, g). Moreover, X is diffeomorphic to the tangent bundle of M, in such a way that u^2 corresponds to the enrgy function on $T(M)$, i.e. the squared norm of a vector.

Automorphisms of $\mathbb{C}^n$, (Andersen-Lempert Inventiones -92

The theory of Automorphisms of $\mathbb{C}^n$ is rather simple when $n = 1$; the only automorphisms are $z \to az + b$. In several variables there are however many more. If we write $z = (z', z_n)$ the group of automorphisms include all *shear maps*

$$z = (z', z_n) \to (z', z_n + f(z')).$$

These shears have the particular property that their Jacobian is identically equal to 1, so they are volume preserving. In a breakthrough paper a few years earlier, Erik Andersen had proved that when $n = 2$,

1. The group of compositions of shears is a proper subgroup of the group of all volume preserving automorphisms.

2. The group generated by shears is, however, dense in the group of all volume preserving automorphisms.

Andersen-Lempert

In the paper of Andersen and Lempert these results are first generalized to arbitrary dimension. The first part of this is very different from the approach of Andersen who exhibited an explicit automorphism that does not belong to the shear group. Second, the results are generalized to the group of all automorphisms, not necessarily volume preserving. One then has to replace the shears by so called *overshears*, maps of the form

$$(z', z_n) \to (z', f(z') + h(z')z_n).$$

The main result is that the group generated by such maps is again a proper dense subgroup of the group of all automorphisms.
In addition to this they prove an important approximation theorem for biholomorphic maps defined on starshaped domains.
The results of this paper have been enormously influential, there are subsequent articles with titles like '30 years of Andersen-Lempert theory'.

Holomorphic functions in infinite dimension. (JAMS -98, JAMS -99, Inventiones -00, Contemporary Math -97

At the end of the 90-ties Laszlo embarked on a daring voyage in the territory of infinite dimensional holomorphy, and $\bar{\partial}$-equations in infinite dimensions. As he remarks in the first (?) paper, this theory had been dormant (or at least not so active) after the sixties or seventies, but motivated but applications to physics and geometry, Laszlo found that it was time to 'revisit' the area. I don't know which applications to geometry that he had in mind, but in the paper from -97 he discusses the problem of complexifying a Lie group, i. e. to find a group corresponding to the complexification of the Lie algebra of a real group. These questions are still very much alive, as can be seen from (again) the example of Mabuchi space of a compact Kahler manifold, which arises as a substitute for the complexification of the symplectic group.

Sharp version of the Ohsawa-Takegoshi extension theorem.(Lempert-B, J of Math Soc Japan -16

I have saved Laszlo's best work till the end – his joint work with me of course. The background to this is the famous Ohsawa-Takegoshi extension theorem. In its simplest form this theorem says that if D is a pseudoconvex domain and V is the intersection of D with a lower dimensional complx subspace of $\mathbb{C}^n$, then any holomorphic function on V can be extended holomorphically to D in such a way that

$$\int_D |H|^2 e^{-\phi} \leq C \int_V |h|^2 e^{-\phi},$$

where C is a universal constant. (Here ϕ is a plurisubharmonic function in D.) The question arises what is the best value of C?

A very particular case is when V is just one point in D, say the origin of $\mathbb{C}^n$. Then, as observed by Ohsawa, the question is equivalent to a lower estimate of the Bergman kernel at the origin in terms of a plurisubharmonic Greens function. In the case when $n = 1$, we have the standard Green's function, and when $\phi = 0$, it was conjectured by Suita that the extremal case was a disk. This (and more) was proved by Blocki, with further extension by Guan and Zhou. Laszlo had the very good idea to connect this to a theorem of myself about plurisubharmonic variation of Bergman kernels. Applying this to the family of domains defined by

$$G(z, 0) < t, \ t < 0$$

the Suita conjecture follows quite easily. In our joint paper we build on this idea to prove similar results for the full OT-extension theorem.

Final words

There are many important results of Laszlo that I have not touched upon at all, e. g. his counterexamples (with Vivas and Darvas) to existence of smooth geodesics in Mabuchi space, moduli of convex domains and his work (with Szoke) on Hilbert fields.

In spite of this I hope my selection illustrates how he moves, seemingly without effort, from one problem to the next, absorbing the existing theory and contributing his own deep original ideas. In this perspective, my last example is a bit atypical as it depends on one clever and simple idea.

Thanks!